# Contents

**Notation**: $\mathbb{N}$ = the set of all natural numbers,
$\mathbb{Z}$ = the set of all integers,
$\mathbb{Q}$ = the set of all rational numbers,
$\mathbb{R}$ = the set of all real numbers,
$\mathbb{C}$ = the set of all complex numbers.

# Chapter 1

# Induction

## 1.1 Mathematical Induction

### 1.1.1 Introduction

Consider the following problems:

1. Prove that the sum of first $n$ positive integers is $\frac{n(n+1)}{2}$.
2. Prove that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.
3. Prove that the number of subsets of an $n$-element set is $2^n$, where $n$ is any positive integer $\geq 1$.
4. Suppose that a bus route has infinitely many stations. The bus stops at the first station. Suppose that, if the bus stops at a station along its route, then it stops at the next station. Show that the bus stops at all stations.
5. Prove that $n < 2^n$ for all positive integers $n$.

If one wants to check the validity of all these statements about positive integers, what should be done? The aim of this chapter is to introduce the student to the technique of Induction and use it to solve a variety of problems. For solving the above problems the proofs using mathematical induction have two parts. First we prove that the statement is true for $n = 1$. Next we assume that $P(n)$ holds for a positive integer $n$ and use this information to show that $P(n+1)$ is true. i.e. if $P(1)$ and $\forall r\ (P(r) \Rightarrow P(r+1))$ are true for the domain of positive integers, then $\forall n P(n)$ is true.
We shall now see an interesting example to illustrate this technique.
Consider that there is an infinite ladder, and we want to know whether we can reach every step on this ladder under the two given conditions.





a) We can reach the first rung of the ladder.

b) If we can reach a particular rung of the ladder, then we can reach the next rung.

Is it possible to show that we can reach every rung?
By (a), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, using (b) we can also reach the second rung. Now making use of (b) again, because we can reach the second rung, we can also reach the third rung. Using similar argument, we can reach the fourth, fifth, sixth rung and so on. For example, after 50 uses of (b), we know that we can reach the $51^{\text{st}}$ rung. But can we conclude that we are able to reach every rung of this infinite ladder? The answer is yes. We can verify this using mathematical induction.

### 1.1.2 Principle of Mathematical Induction

A proof using mathematical induction has two parts:

- A base step where we show that $P(1)$ is true;
- An inductive step where we show that for all positive integers $r$, if $P(r)$ is true, then $P(r+1)$ is true.

**Principle of Mathematical Induction**: To prove that $P(n)$ is true for all positive integers $n$ where $P(n)$ is a propositional function, we follow a two step procedure:
**BASE STEP** We verify that $P(1)$ is true.
**INDUCTIVE STEP** We show that the conditional statement
$P(r) \rightarrow P(r+1)$ is true for all positive integers $r$.
The assumption that $P(r)$ is true is called the **inductive hypothesis**.
Once both the above steps are completed, we have shown that $P(n)$ is true for all positive integers.

**Remark 1.1** *Sometimes a statement has to be proved for all positive integers $n \geq n_0$. In this case the base step involves verifying that $P(n_0)$ is true.*

We shall use mathematical induction to solve all the above problems.

### 1.1.3 Illustrative Examples

1. The sum of first $n$ positive integers is $\frac{n(n+1)}{2}$.
   **Solution:** Let $P(n)$ be the proposition that the sum of the first $n$ positive integers is $n(n+1)/2$. We must show that $P(1)$ is true and that the conditional statement $P(k) \Rightarrow P(k+1)$ is true for $k = 1, 2, 3, \ldots$
   **BASE STEP** $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$.
   **INDUCTIVE STEP** For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer $k$. That is, we assume that
   $$1 + 2 + \ldots + k = \frac{k(k+1)}{2} \qquad (1)$$
   Under this assumption, it must be shown that $P(k+1)$ is true, namely, that
   $$1 + 2 + \ldots + k + (k+1) = = \frac{(k+1)(k+2)}{2}$$
   is also true.
   When we add $k+1$ to both sides of the equation (1) we get,
   $$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$
   Since we have completed both, the BASE step and the inductive step we have shown that $P(n)$ is true for all positive integers $n$.

2. Prove that for every positive integer $n$, $n^3 - n$ is divisible by 3.
   **Solution:** Let $P(n)$ be the proposition that for every positive integer $n$, $n^3 - n$ is divisible by 3. We must show that $P(1)$ is true and



   that the conditional statement $P(k) \Rightarrow P(k+1)$ is true for $k = 1, 2, 3, \ldots$
   **BASE STEP** $P(1)$ is true, because $1^3 - 1 = 0$ is divisible by 3.
   **INDUCTIVE STEP** For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer $k$. That is, we assume that,
   $$\text{For every positive integer } k,\ k^3 - k \text{ is divisible by } 3. \qquad (2)$$
   Under this assumption, it must be shown that $P(k+1)$ is true, namely, that
   $$3|[(k+1)^3 - (k+1)]$$
   is also true.
   $$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= k^3 - k + 3(k^2 + k). \end{aligned}$$
   Using (2) we know that $k^3 - k$ is divisible by 3 for every positive integer $k$. Moreover $3(k^2 + k)$ is also a multiple of 3 for every positive integer $k$. Thus $k^3 - k + 3(k^2 + k)$ is divisible by 3.
   Since we have completed both, the base step and the inductive step we have shown that $P(n)$ is true for all positive integers $n$.

3. Prove that for every positive integer $n$, the number of subsets of an $n$-element set is $2^n$.
   **Solution:** Let $P(n)$ be the proposition that for every positive integer $n$, the number of subsets of an $n$-element set is $2^n$. We must show that $P(1)$ is true and that the conditional statement $P(k) \Rightarrow P(k+1)$ is true for $k = 1, 2, 3, \ldots$
   Let $S_n = \{a_1, a_2, \ldots, a_n\}$ be a set containing $n$ elements.
   **BASE STEP** $P(1)$ is true since the set $S_1$ has only two subsets viz. $\phi$ and $\{a_1\}$.
   **INDUCTIVE STEP** For the inductive hypothesis we assume that

$P(k)$ holds for an arbitrary positive integer $k$. That is, we assume that, for every positive integer $k$,

$$\text{The number of subsets of a } k \text{ element set is } 2^k. \tag{3}$$

Under this assumption, it must be shown that $P(k+1)$ is true, i.e. we shall prove that the number of subsets of a $(k+1)$-element set is $2^{k+1}$.
Consider $S_{k+1} = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$ be a set containing $k+1$ elements. The subsets of $S_{k+1}$ either contain $a_{k+1}$ or do not contain it. The subsets not containing $a_{k+1}$ are precisely the subsets of $S_k$. Using (3), we get that this number is $2^k$. The subsets not containing $a_{k+1}$ are precisely the subsets of $S_k \cup \{a_{k+1}\}$ . Again using (3) we get that this number is $2^k$. Thus the total number of subsets of $S_{k+1}$ equals $2^k + 2^k = 2^k(1+1) = 2^{k+1}$.
Since we have completed both, the base step and the inductive step, we have shown that $P(n)$ is true for all positive integers $n$.

4. There are infinitely many stations on a train route. The train stops at the first station. Suppose that if the train stops at a station, then it stops at the next station. Show that the train stops at all stations.
**Solution:** Let $P(n)$ be the proposition that for every positive integer $n$, the train stops at the $n^{\text{th}}$ station. We must show that $P(1)$ is true and that the conditional statement $P(k) \Rightarrow P(k+1)$ is true for $k = 1, 2, 3, \ldots$
**BASE STEP** $P(1)$ is true since it is given that the train stops at the first station.
**INDUCTIVE STEP** For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer $k$. That is, we assume that, for every positive integer $k$,

$$\text{The train stops at the } k^{\text{th}} \text{ station.} \tag{4}$$

Under this assumption, it must be shown that $P(k+1)$ is true, i.e. we shall prove that the train stops at the $(k+1)^{\text{th}}$ station.
It is known that if the train stops at a station, then it stops at the next



station. Thus using (4) and the given information, if the train stops at the $k^{\text{th}}$ station then it stops at the $(k+1)^{\text{th}}$ station.
Since we have completed both, the base step and the inductive step we have shown that $P(n)$ is true for all positive integers $n$.

5. Show that $n < 2^n$ for all positive integers $n$.
**Solution:** Let $P(n)$ be the proposition that for every positive integer $n$, $n < 2^n$. We must show that $P(1)$ is true and that the conditional statement $P(k) \Rightarrow P(k+1)$ is true for $k = 1, 2, 3, \ldots$.
**BASE STEP** $P(1)$ is true, because $1 < 2$.
**INDUCTIVE STEP** For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer $k$. That is, we assume that,

$$\text{For every positive integer } k,\ k < 2^k. \tag{5}$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$(k+1) < 2^{k+1}$$

is also true. We note that $2^{k+1} = 2^k + 2^k$. Using (5), we get that $2^k > k$. Thus

$$2^k + 2^k \quad > \quad k + k \geq k + 1.$$

Thus we have shown that $(k+1) < 2^{k+1}$ for all positive integers $n$.
Since we have completed both, the base step and the inductive step, we have shown that $P(n)$ is true for all positive integers $n$.

## 1.2 Strong Induction

In the earlier section we have studied Mathematical Induction and used it to solve a variety of problems. In this section we will introduce another form of mathematical induction, called strong induction which can often be used when we cannot prove a result using mathematical induction. Consider the following examples:

1. Show that if $n$ is an integer greater than 1, then $n$ can be written as a product of primes.

2. At the beginning Hari can run either one mile or two miles. If he can always run two more miles, once he has run a specified number of miles, then prove that he can run $n$ miles, where $n$ is any natural number.

3. Suppose that a chocolate bar consists of $n$ squares of chocolates as usual arranged in a rectangular pattern. The bar can be broken along the vertical or horizontal line separating the squares. Assuming that only one cut along either the horizontal or vertical cut can be made at a time, show that the number of breaks you must successively make to break the bar into $n$ separate squares is $n-1$.

You should try to solve each of these problems using the First form of Mathematical induction that has been studied in the earlier section and analyze the difficulties encountered in proving the statements.

### 1.2.1 Statement of Strong Induction

**Aim:** To prove that $P(n)$ is true for all positive integers $n$, where $P(n)$ is a propositional function. We have to perform two steps to prove the proposition.
**Base Step:** We begin by proving that $P(1)$ is true.
**Inductive Step:** Next we show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \Rightarrow P(k+1)$ is true for all positive integers $k$.

1. Show that if $n$ is an integer greater than 1, then $n$ can be written as a product of primes.
   **Solution:** $P(n)$ is a proposition that $n$ can be written as a product of primes.
   **BASE STEP**: $P(2)$ is true since 2 itself is a prime.
   **INDUCTIVE STEP**: For the inductive hypothesis we assume that $P(k)$ holds for all positive integers $j$ with $2 \leq j \leq k$. That is, we assume that, for every positive integer $j$ where $j \leq k$, $j$ can be



   written as a product of primes. We have to prove that $P(k+1)$ is true.

   To prove that $(k+1)$ can be written as a product of primes.

   Case 1: Suppose $k+1$ is itself a prime, then we are through.
   Case 2: Suppose $k+1$ is composite, then $k+1$ can be written as a product of two integers $u$ and $v$ such that $k+1 = uv$ with $2 \leq u, v < (k+1)$. By inductive hypothesis $u$ and $v$ can be written as a product of primes and hence $k+1$ is written as a product of primes. The factorization of $k+1$ will contain the primes in the factorization of $u$ and $v$.

2. At the beginning Hari can run either one mile or two miles. If he can always run two more miles, once he has run a specified number of miles, then prove that he can run $n$ miles, where $n$ is any natural number.
   **Solution:** We will show that Hari can run any number of miles using the method of strong induction.
   **BASE STEP**: $P(1)$ is true since Hari can run one mile at the beginning.
   **INDUCTIVE STEP**: The inductive hypothesis implies that Hari can travel $j$ miles where $1 \leq j \leq k$. Assuming the inductive hypothesis we want to show that if Hari can reach each of the first $j$ miles where $1 \leq j \leq k$ then Hari can reach the $(k+1)^{th}$ mile. We also know that Hari can reach two miles. For $k > 2$, we wish to prove that $P(k+1)$ is true. By inductive hypothesis Hari can reach up to $k-1$ miles. Now Hari can travel two miles from the destination that he has already reached and hence he can travel $k+1$ miles.
   Thus we have proved that when Hari can run one mile or two miles and when he can reach all of the first $k$ miles then he can reach the $(k+1)^{th}$ mile.

3. Suppose that a chocolate bar consists of $n$ squares of chocolates as usual arranged in a rectangular pattern. The bar can be broken along the vertical or horizontal line separating the squares. Assuming that only one cut along either the horizontal or vertical cut can be made at

a time, show that the number of breaks you must successively make to break the bar into $n$ separate squares is $n-1$.
**Solution:** Let $P(k)$ be the proposition that at most $k-1$ steps are required to break a chocolate bar consisting of $k$ pieces.
**BASE STEP**: $P(1)$ is true since we need not do anything if the chocolate bar consists of just one piece. i.e. 0 cuts are required.
**INDUCTIVE STEP**: Now we shall assume that $P(k)$ is true for $1 \leq k \leq (n-1)$. We have to prove that $P(n) = n-1$. You may break the chocolate bar consisting of two pieces into two pieces of size $k_1$ and $k_2$ such that $k_1 + k_2 = n$. By the inductive hypothesis $P(k_1) = k_1 - 1$ and $P(k_2) = k_2 - 1$. Thus the bar consisting of $n$ pieces has been broken $1 + (k_1 - 1) + (k_2 - 1) = n - 1$ times.

The Principle of Mathematical Induction and Strong Induction are a very useful method of proof. The validity of both, the Principle of Mathematical Induction and Strong Induction follows from a fundamental axiom of the set of natural numbers known as the Well Ordering Property which independently can be used to prove many useful results especially in the theory of numbers.
We now state the two principles.
**(i) Well Ordering Principle :** Any non-empty subset of non-negative integers has a smallest element.

In other words, if $S$ is a non-empty subset of non-negative integers then there is an $s \in S$ such that $s \leq a$ for every $a$ in $S$.

**(ii) Principle of Mathematical Induction :** If a subset $S$ of positive integers contains 1, and contains $n+1$ whenever it contains $n$, then $S$ contains all the positive integers.

**(iii) Strong Form of Induction :** If a subset $S$ of positive integers contains 1, and if $n$ is a positive integer such that $1, 2, \ldots, n \in S$ then $n+1$ also belongs to $S$. i.e. $S$ contains all the positive integers.

**Remark 1.2** Note that the Well Ordering Principle is not true for the set of integers.



### 1.2.2 Exercises

1. Prove that for every positive integer $n$,
   $1.2 + 2.3 + \ldots + n(n+1) = n(n+1)(n+2)/3$.

2. Prove that for every positive integer $n$,
   $1.1! + 2.2! + \ldots + n.n! = (n+1)! - 1$.

3. Prove that $2|(n^2 + n)$, whenever $n$ is a positive integer.

4. Prove that $3|(n^3 + 2n)$, whenever $n$ is a positive integer.

5. Prove that $n^2 - 1$ is divisible by 8, whenever $n$ is an odd positive integer.

6. Prove that $9|[4^n + 15n - 1]$, whenever $n$ is a positive integer.

7. Prove that a set with $n$ elements has $n(n-1)/2$ subsets containing exactly two elements whenever $n$ is an integer greater than or equal to 2.

8. Suppose that $A$ and $B$ are square matrices with the property that $AB = BA$. Show that $AB^n = B^n A$ for every positive integer $n$.

9. Prove that $(1+x)^n > 1 + nx$, where $x > -1, x \neq 0$, and $n = 2, 3, \ldots$.

10. Show that $1 + n\sqrt{2^{n-1}} < 2^n$ for all positive integers $n \geq 2$.

11. Show that $n^2 < 2^n$ for all positive integers $n \geq 5$.

12. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$ then $\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j$.
    Formulate and prove a similar result for intersection of sets.

13. Show that every positive integer $n$ can be written as a sum of distinct powers of two.

14. A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with $n$ pieces.

15. Find the amounts of money that can be formed using just coins of 2 rupees and coins of 5 rupees.

16. Show that if $a_1, a_2, \ldots, a_n$ are $n$ distinct real numbers, exactly $n-1$ multiplications are used to compute the product of these $n$ numbers, no matter how parentheses are inserted into their product.

17. Suppose you begin with a pile of $n$ stones and split this pile into $n$ piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have $r$ and $s$ stones in them, respectively, you compute $rs$. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n-1)/2$.

## Appendix

### $\sqrt{2}$ is irrational.

We need to show that $\sqrt{2} \neq \frac{a}{b}$ for all integers $a$ and $b$. That is, $2 \neq \dfrac{a^2}{b^2}$ for all integers $a$ and $b$. We use the Strong version of Mathematical Induction: Let $P(n)$ denote a property that holds for integer n. Suppose that $P(n_0)$ is true; and for all $k \geq n_0, P(m)$ is true for all $m$ with $n_0 \leq m \leq k$ implies $P(k+1)$ is true. Then $P(n)$ is true for all $n \geq n_0$.

In our proof here we will take $n_0 = 1$, for we know that $2 \neq \dfrac{1}{b^2}$ for all integers $b$. Suppose that $2 \neq \dfrac{a^2}{b^2}$, for all $a$, where $1 \leq a \leq n$. Suppose $2 = \dfrac{(n+1)^2}{b^2}$. Then $(n+1)^2 = 2b^2$ so that $(n+1)^2$ is even and hence $n+1$



is even, say $n+1 = 2a'$. Then $b^2 = 2a'^2$, or $2 = \dfrac{b^2}{(a')^2}$; a contradiction to the induction hypothesis since $b < n+1$. So $2 \neq (n+1)^2/b^2$, and hence the statement holds for $n+1$. By Strong Induction, the statement holds for all $n$. That is, $2 \neq \dfrac{a^2}{b^2}$ for all integers $a$ and $b$. Hence $\sqrt{2}$ is not rational.

# Chapter 2

# Divisibility of Integers

## 2.1 Divisibility of Integers

In this section, we see some elementary properties of integers. Many of the proofs depend on two principles (i) Well Ordering Principle, (ii) Principle of Mathematical Induction.

**Definition 2.1** An integer $b$ is said to be divisible by a non-zero integer $a$ if there is an integer $x$ such that $b = ax$, and we then write $a|b$. In case $b$ is not divisible by $a$ we write $a \nmid b$. The property $a|b$ may also be expressed by saying that '$a$ divides $b$' or '$a$ is a divisor of $b$' or '$b$ is a multiple of $a$'.

Note that 7 divides $14$ as $14 = 7 \times 2$. But 7 does not divide $13$. We may write this as $7|14$ but $7 \nmid 13$.

**Theorem 2.1** $(i)$ If $a|b$ then $a|bc$ for any integer $c$.

$(ii)$ If $a|b$ and $b|c$ then $a|c$.

$(iii)$ If $a|b$ and $a|c$ then $a|bx + cy$ for any integers $x$ and $y$.

$(iv)$ If $a|b$, $b \neq 0$, then $|a| \leq |b|$.

$(v)$ If $a|b$ and $b|a$ then $a = \pm b$.

$(vi)$ If $m \neq 0$ then $a|b$ if and only if $ma|mb$.

**Proof.**

$(i)$ If $a|b$ then $b = aq$, where $q$ is an integer. Hence, $bc = a(qc)$ for any integer $c$. Hence, $a|bc$.

$(ii)$ If $a|b$ and $b|c$ then $b = aq$ and $c = bq_1$ where $q, q_1$ are integers. Thus, $c = (aq)q_1 = a(qq_1)$. Hence, $c|a$.





$(iii)$ If $a|b$ and $a|c$ then $b = aq, c = aq_1, q, q_1 \in \mathbb{Z}$. Hence, $bx + cy = a(qx + q_1y)$. Hence, $a|bx + cy$.

$(iv)$ If $a|b$, $b \neq 0$, then $b = aq, q \neq 0$. Hence, $|b| = |aq| = |a||q|$. As $q \neq 0, |q| \geq 1$, hence $|a| \leq |b|$.

$(v)$ If $a|b$ and $b|a$ then $|a| \leq |b|$ and $|b| \leq |a|$. Hence, $a = \pm b$.

$(vi)$ If $a|b$ then $b = aq$. Suppose $m \neq 0$ then $mb = (ma)q$. Hence, $ma|mb$. Conversely, $ma|mb$ then $mb = maq$. Since, $m \neq 0$, we get $b = aq$, that is, $a|b$.

**Theorem 2.2 (Division Algorithm)** Given integers $a$ and $b$ with $a \neq 0$, there exist unique integers $q$ and $r$ such that

$$b = qa + r, 0 \leq r < |a|.$$

If $a \nmid b$ then $r$ satisfies the stronger inequality $0 < r < |a|$.

**Proof.** Consider, $S = \{b - ak | b - ak \geq 0,\ k \in \mathbb{Z}\}$. Clearly, $b + |ab| \in S$. Hence, $S$ is non-empty. By well ordering principle, $S$ has a least element, say $b - aq = r$. If $r \geq |a|$ then $0 < r - |a| < r$ and $r - |a| \in S$, a contradiction.

Next we prove the uniqueness of $q$ and $r$. Suppose $b = aq_1 + r_1$ and also $b = aq_2 + r_2$ with $0 \leq r_1 < |a|,\ 0 \leq r_2 < |a|$. If $r_1 \neq r_2$, let $r_1 < r_2$. Then $0 < r_2 - r_1 \leq r_2 < |a|$. Now, $r_2 - r_1 = a(q_1 - a_2)$. Thus $a|(r_2 - r_1)$. As $r_2 - r_1 > 0$ (v) of the theorem 1 implies that $|a| \leq (r_2 - r_1)$, a contradiction. Hence $r_2 = r_1$ and so $q_2 = q_1$.

**Example 2.1** Show that the square of any integer is of the form $4k$ or $8k + 1$.

**Solution.** By division algorithm (take $a = 2$), any integer $b$ is representable as $2q$ or $2q + 1$. If $b = 2q$, then and $b^2 = 4q^2$ i.e. $b^2$ is of the form $4k$. If $b = 2q + 1$, then $b^2 = 4q^2 + 4q + 1 = 4q(q + 1) + 1$. Since $q(q + 1)$ is divisible by 2, we get that $b^2$ is of the form $8k + 1$.

**Example 2.2** Show that the square of any integer is of the form $9k$ or $3k + 1$.

**Solution.** By division algorithm (take $a = 3$), any integer $b$ is representable as $3q$ or $3q \pm 1$. If $b = 3q$, then and $b^2 = 9q^2$ i.e. $b^2$ is of the form $9k$. If $b = 3q \pm 1$, then $b^2 = 9q^2 \pm 6q + 1 = 3k + 1$.

**Example 2.3** Find all integers $n$ such that $n^2 + 1$ is divisible by $n + 1$.
**Solution.** Let $n$ be an integer such that $n + 1 | n^2 + 1$. Note that $n + 1 | (n + 1)(n - 1)$ i.e. $n + 1 | n^2 - 1$. Hence, $n + 1 | (n^2 + 1) - (n^2 - 1)$ i.e. $n + 1 | 2$. Hence, $n + 1 = \pm 1, \pm 2$. Hence, $n = -3, -2, 0, 1$.

## 2.2 Greatest Common Divisor

**Definition 2.2** An integer $d$ is called a common divisor of $a$ and $b$ in case $d|a$ and $d|b$. If atleast one of $a$ and $b$ is not equal to 0, the greatest among their common divisors is called the **greatest common divisor** of $a$ and $b$ and is denoted by $(a, b)$.

In other words, a positive integer $g$ is greatest common divisor of $a$ and $b$ if and only if the following two conditions are satisfied: (i)$g|a$ and $g|b$ and (ii) if $d|a$ and $d|b$ then $d|g$.

Note that $1, 2, 5, 10$ are common divisors of $20$ and $50$ and $10$ is the greatest common divisor of $20$ and $50$.

**Definition 2.3** Two integers $a$ and $b$ are said to be relatively prime (co-prime) if $(a, b) = 1$.

For example, 10 and 21 are relatively prime integers. Any two consecutive integers are relatively prime.

**Theorem 2.3 (Bezout's Theorem)** If $g = (a, b)$, then there exist integers $x_0$ and $y_0$ such that $g = ax_0 + by_0$.
**Proof.** Consider, $S = \{ax + by | x, y \in \mathbb{Z},\ ax + by > 0\ \}$. $S$ is non-empty as $a^2 + b^2 \in S$. By well ordering principle, $S$ has a smallest element, say $ax_0 + by_0 = g$. If $d|a$ and $d|b$ then $d|ax_0 + by_0$ i.e. $d|g$. Suppose $g \nmid a$ then $a = gq + r,\ 0 < r < g$. Hence,

$$r = a - gq = a(1 - qx_0) + b(-qy_0) \text{ and } r \in\ S.$$



But $g$ is the smallest element of $S$, a contradiction. Hence, $g|a$. Similarly $g|b$. Hence, $g$ can be written as $ax_0 + by_0$.

**Theorem 2.4 (The Euclidean algorithm)** Given integers $b$ and $c > 0$, we make a repeated application of the division algorithm to obtain a series of equations

$$\begin{aligned}
b &= cq + r_1, \qquad 0 < r_1 < c,\\
c &= r_1 q_1 + r_2, \qquad 0 < r_2 < r_1,\\
&\vdots\\
r_{j-2} &= r_{j-1} q_{j-1} + r_j, \qquad 0 < r_j < r_{j-1},\\
r_{j-1} &= r_j q_j.
\end{aligned}$$

The greatest common divisor $(b, c)$ of $b$ and $c$ is $r_j$, the last non-zero remainder in the division process.

Moreover, if $(b, c) = bx_0 + cy_0$ then the values of $x_0$ and $y_0$ can be obtained by eliminating $r_{j-1}, ..., r_2, r_1$ from the set of equations.

**Remark 2.1**
1. We note that there is no loss of generality in assuming that $c$ is positive for $(b, c) = (b, -c) = (-b, c) = (-b, -c)$.
2. Note that the values of $x_0$ and $y_0$ are not unique. For example, $1 = -1(2) + 1(3)$ and $1 = 2(2) - 3$.

**Example 2.4** Find $\gcd$ of 704 and 407. Also find $x_0$, $y_0$ such that

$$(704, 407) = 704x_0 + 407y_0.$$

**Solution.** We have

$$\begin{aligned}
704 &= 407 + 297 && (1)\\
407 &= 297 + 110 && (2)\\
297 &= 2(110) + 77 && (3)\\
110 &= 77 + 33 && (4)\\
77 &= 2(33) + 11 && (5)\\
33 &= 3(11). && (6)
\end{aligned}$$

The last nonzero remainder is 11, hence $(704, 407) = 11$. Now to find $x_0$, $y_0$ such that $11 = 704x_0 + 407y_0$, we write (5) $11 = 77 - 2(33)$. Substituting for 33 from (4),

$$11 = 77 - 2(110 - 77) = 3(77) - 2(110).$$

Substituting for 77 from (3),

$$11 = 3(297 - 2(110)) - 2(110) = 3(297) - 8(110).$$

Substituting for 110 from (2),

$$11 = 3(297) - 8(407 - 297) = 11(297) - 8(407).$$

Substituting for 297 from (1),

$$11 = 11(704 - 407) - 8(407) = 11(704) - 19(407)$$

so that $x_0 = 11, y_0 = -19$. We can take $x_0 = 11 + 407k$ and $y_0 = -19 - 704k, k \in \mathbb{Z}$. Note that there are many possible values for $x_0$ and $y_0$.

**Example 2.5** Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number $n$.

**Solution.** We want to show that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number $n$, that is, we should show that $(21n+4, 14n+3) = 1$ for every natural number $n$. Now,

$$\begin{aligned} 21n+4 &= 14n+3+7n+1 \\ 14n+3 &= 2(7n+1)+1 \\ 7n+1 &= 1(7n+1) \end{aligned}$$

Hence, by Euclidean algorithm, $(21n+4, 14n+3) = 1$. Hence, the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number $n$.



**Definition 2.4** Let $a$, $b$ be non-zero integers. An integer $m$ is called a common multiple of $a$ and $b$ in case $a|m$ and $b|m$. The least among of the positive common multiples is called the **least common multiple** of $a$ and $b$ and is denoted by $[a, b]$.

In other words, a positive integer $l$ is least common multiple of $a$ and $b$ if and only if the following two conditions are satisfied: (i) $a|l$ and $b|l$ and (ii) if $a|m$ and $b|m$ then $l|m$.

**Example 2.6** The sum of two positive integers is 52 and their l.c.m. is 168. Find the numbers.

**Solution.** Let the positive integers be $a$ and $b$ and $a \leq b$. Let $d = (a, b)$ so that $a = dm$, $b = dn$ where $(m, n) = 1$. Thus (i) $a + b = d(m+n) = 52 = 4 \times 13$ and (ii) l.c.m. of $a, b = dmn = 168 = 4 \times 2 \times 7 \times 3$. But $((m+n)d, mnd) = d$, since $(m, n) = 1$. Hence by (i) and (ii), $d = 4$. So $m + n = 13$ and $mn = 42$, which give $m = 6, n = 7$. Hence $a = dm = 24$, $b = dn = 28$.

## 2.3 Primes

**Definition 2.5** An integer $p > 1$ is called a prime number, or a prime, if it has no divisor $d$ such that $1 < d < p$. If an integer is not a prime then it is called a composite number.

Note that $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$ are prime numbers, while $4, 6, 8, 15, 20$ are composite numbers.

**Theorem 2.5 (Euclid's Lemma)** If $(a, m) = 1$ and $m|ab$ then $m|b$.

**Proof.** Since $(a, m) = 1$, there exist integers $x, y$ such that $ax + my = 1$ Hence, $abx + mby = b$. Since, $m|ab$ and $m|m$ we get $m|abx + mby$. Hence, $m|b$.

**Corollary 2.1** If $p$ is a prime and $p|ab$ then $p|a$ or $p|b$.

**Proof.** If $p|a$ then we are done. Otherwise, $p \nmid a$. Hence, $(p, a) = 1$. Since, $p|ab$ and $(p, a) = 1$, by Euclid's lemma, we get that $p|b$.

Euclid's lemma can also be generalised for the product of $n$ integers. This proof is based on Principle of Mathematical Induction. Induction is on the number of terms occurring in the product.

**Corollary 2.2** If $p$ is a prime such that $p|a_1a_2\cdots a_n$, then $p$ divides at least one factor $a_i$ of the product.
**Proof.** If $n = 2$, then $p|a_1a_2$. Hence, $p|a_1$ or $p|a_2$. Assume that the result holds for the product of $n$ integers. Suppose $p|a_1\cdots a_na_{n+1}$, then $p|(a_1\cdots a_n)a_{n+1}$. Hence, either $p|a_1a_2\cdots a_n$ or $p|a_{n+1}$. If $p|a_{n+1}$ then we are done. Otherwise, $p|a_1a_2\cdots a_n$. Hence, by induction hypothesis, $p|a_i$ for some $i$, $1\le i\le n$. Hence, by principle of mathematical induction, we get the result.

**Example 2.7** Show that, if $p$ is a prime and $r$ is a positive integer such that $\le r < p$ then $p\nmid r!$.
**Solution.** If $p\mid r!$ then by Euclid's lemma, $p\mid k$ for some $k, 1\le k\le p$, a contradiction. Hence, $p\nmid r!$.

**Example 2.8** Show that, if $p$ is a prime then $p|\binom{p}{r}$ for $0 < r < p$.
**Solution.** Note that $\binom{p}{r} = \dfrac{p!}{r!(p-r)!}$. Hence, $p! = \binom{p}{r}r!(p-r)!$. Note that $p|p!$ and as, $1\le r\le p-1$, $p\nmid |r!$ and $p\nmid (p-r)!$. Hence, using Euclid's lemma , we get $p|\binom{p}{r}$.

**Example 2.9** Prove that if $p$ is a prime, then $\sqrt{p}$ is an irrational number.
**Solution.** Suppose $\sqrt{p}$ is a rational number, say $\dfrac{a}{b}$, where $a, b$ are relatively prime integers. Hence, $p = \dfrac{a^2}{b^2}$. Thus, $pb^2 = a^2$. Hence, $p|a^2$. By Euclid's lemma, $p|a$. Hence, $p^2|a^2$. But $a^2 = pb^2$. Hence, $p|b^2$. Hence, $p|b$, a contradiction. Hence, $\sqrt{p}$ is an irrational number.

**Remark 2.2 (Fundamental theorem of Arithmetic)** Every positive integer greater than 1 can be expressed as product of primes in a unique way except for the order of the prime factors.

Using principle of Mathematical induction (strong form), we can prove that every positive integer greater than 1 can be expressed as product of primes. The uniqueness of the factorization follows from Euclid's lemma.



**Notes**

1. A number $n = p_1^{a_1}p_2^{a_2}\ldots p_r^{a_r}$ is a perfect square if and only if each of $a_1, a_2, \ldots, a_r$ is even. If $n = p_1p_2\ldots p_r$ (i.e. if each of $a_1, a_2, \ldots, a_r$ is equal to 1) then $n$ is called a square-free integer.

2. If $a, b$ are positive integers such that $ab = c^k$ and $(a, b) = 1$, then $a$ and $b$ are both perfect $k$-th powers.

**Remark 2.3** There are infinitely many primes.

**Example 2.10** Given any positive integer $n$, show that there exist $n$ consecutive composite integers.
**Proof.** Consider the integers

$$(n+1)!+2, (n+1)!+3, \ldots, (n+1)!+n, (n+1)!+(n+1).$$

Every number of the sequence is a composite number because $k$ divides $(n+1)!+k$ if $2\le k\le (n+1)$. Hence, we get $n$ consecutive composite numbers.

**Example 2.11** If $p$ is a prime greater than 3 then show that $2p+1$ and $4p+1$ can not be primes simultaneously.
**Solution.** Since $p$ is a prime greater than 3, $p$ is either of the type $3k+1$ or $3k+2$. If $p$ is of the type $3k+1$ then $2p+1 = 2(3k+1)+1 = 6k+3 = 3(2k+1)$. Hence, $3|2p+1$ and $2p+1$ can not be a prime. Similarly, if $p$ is of the type $3k+2$ then 3 divides $4p+1$ and it can not be a prime.

## Exercise Set - 2.1

1. Prove that no integer in the sequence 11, 111, 1111, ... is a perfect square.

2. Show that the square of any integer is of the form $9k$ or $3k+1$.

3. Show that for any positive integer $m$, $(ma, mb) = m(a, b)$.

4. Show that if $d|a$ and $d|b$ and $d > 0$ then $(\frac{a}{d}, \frac{b}{d}) = \dfrac{(a, b)}{d}$.

5. Show that if $(a, m) = (b, m) = 1$, then $(ab,\ m) = 1$.

6. If $a = \prod_{i=1}^{k} p_i^{\alpha_i}, \quad b = \prod_{i=1}^{k} p_i^{\beta_i}$ then show that

$$(a,\ b) = \prod_{i=1}^{k} p_i^{\min(\alpha_i,\beta_i)}, \quad [a,\ b] = \prod_{i=1}^{k} p_i^{\max(\alpha_i,\beta_i)}.$$

7. If $m > 0$ then show that

$$[ma, mb] = m[a, b] \text{ and } [a,\ b](a,\ b) = |ab|.$$

8. Show that $(a, b) = (b, a) = (a, -b) = (a, b + ax)$ for any integer $x$.

9. By using Euclidean algorithm find the $\gcd$ of (i) 7645 and 2872 and (ii) 963 and 657. Also express the $\gcd$ as the linear combination of the given numbers.

10. Find $(a^{2^m} + 1, a^{2^n} + 1)$. Hence, show that there are infinitely many primes. (Due to Polya.)

11. Let $a, b, c$ be integers such that $(a, b) = 1,\ c > 0$. Prove that there is an integer $x$ such that $(a + bx, c) = 1$.

12. Suppose $m, n$ are integers and $m = n^2 - n$. Then show that $m^2 - 2m$ is divisible by 24.

13. A printer numbers the pages of a book starting with 1 and uses 3189 digits in all. How many pages does the book have?

14. Let $p > 3$ be an odd prime. Show that the numerator of $\sum_{k=1}^{p-1} \frac{1}{k}$ is divisible by $p$.

15. Prove that if $n \geq 4$ then $n, n + 2, n + 4$ can not all be primes.



16. If $2 = p_1 < p_2 < \ldots < p_n$ where $p_i$ are primes, show that the number $p_1 p_2 \cdots p_n + 1$, can never be a perfect square.

17. Prove that, if $n > 4$, then the number $1! + 2! + 3! + \cdots n!$, is never a square.

18. The $\gcd$ of two positive integers is 81 and their l.c.m. is 5103. Find the numbers.

19. Prove that there are infinitely many positive integers $a$ such that $2a$ is a square, $3a$ is a cube and $5a$ is a fifth power.

20. If $a, b$ are positive integers such that the number $\frac{a+1}{b} + \frac{b+1}{a}$ is also an integer, then prove that $\gcd(a, b) \leq \sqrt{a + b}$.

21. Prove that if $p$ is a prime, then $\sqrt[n]{p}$ is an irrational number, where $n$ is an integer greater than or equal to 2.

22. Show that if $p$ and $8p - 1$ are primes then $8p + 1$ is composite.

23. If $2^n - 1$ is a prime, show that $n$ is a prime.

24. Find all integers $x$ and $y$ such that $(x, y) = 8$ and $[x, y] = 64$.

25. If $(a, b) = [a, b]$ then show that $a = \pm b$.

26. Show that if $n$ is an odd integer, then $16 | n^4 + 4n^2 + 11$.

27. Find all integers which leave remainder 1 when divided by 3, remainder 2 when divided by 4, . . . , remainder 8 when divided by 10. $x$ is a solution if and only if $x + 2$ is divisible by $3, 4, \ldots, 10$.

28. Show that an integer $n > 1$ is a composite number if and only if it has a prime divisor $d$ such that $d \leq \sqrt{n}$.

29. Find the $\gcd$ of 3645 and 2357. Also find $x_0$, $y_0$ such that

$$(3645, 2357) = 3645x_0 + 2357y_0.$$

30. Find the given integers $a, b$ use the Euclidean algorithm to find the $\gcd$ and express it as a linear combination of the given numbers.

| | | |
|---|---|---|
| (i) $143, 227$ | (ii) $306, 657$ | (iii) $272, 1479$ |
| (iv) $216, 771$ | (v) $30031, 16579$ | (vi) $56, 72$ |
| (vii)$119, 272$ | (viii) $595, 252$ | (ix) $1769, 2378$ |
| (x) $5291, 4514$ | (xi) $7234, 3476$ | |

31. Find integers $x, y$, if they exist, such that

| | |
|---|---|
| (i) $35x + 49y = 3$ | (ii) $35x + 49y = 21$ |
| (iii) $12x + 7y = 93$ | (iv) $918x + 534y = 424$. |

32. Prove that two integers $a$ and $b$ are relatively prime if and only if there exist integers $\lambda$ and $\mu$ such that $a\lambda + b\mu = 1$.

## 2.4 Relations

We use the notion of relation in our everyday life quite often. For example, Anil is a brother of Sunita or Gopal is a son of Govind. We use the notion of relation in Mathematics as well. For example, we say that 2 is less than 3 or 5 is not equal to 7 or 3 divides 6.

In this section, we define the notion of relation in the context of Mathematics. We discuss the notion of an ordered pair, cartesian product and relation. We also study various types of relations. Intuitively, an ordered pair consists of two elements, say $a$ and $b$, in which one of them, say $a$, is designated as the first element and the other as the second element. An ordered pair is denoted by $(a, b)$. Two ordered pairs $(a, b)$ and $(c, d)$ are equal if and only if $a = c$ and $b = d$. Thus, $(2, 3)$ and $(3, 2)$ are not equal.

**Remark 2.4** An ordered pair $(a, b)$ can be defined rigorously by

$$(a, b) \equiv \{\{a\}, \{a, b\}\}.$$

From this definition, we can prove that

$$(a, b) = (c, d) \text{ implies that } a = c \text{ and } b = d.$$



**Definition 2.6** Let $A$ and $B$ be nonempty sets. The cartesian product of $A$ and $B$ is defined as the set of all ordered pairs $(x, y)$ where $x$ is an element of $A$ and $y$ is an element of $B$.

The cartesian product of $A$ and $B$ is denoted by $A \times B$. Thus,

$$A \times B = \{(x, y) | x \in A, y \in B\}.$$

If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ then

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}.$$

**Definition 2.7** Let $A$ and $B$ be nonempty sets. A relation $R$ from $A$ to $B$ is a subset of $A \times B$. If $(a, b) \in R$ then we say that $a$ is related to $b$. If $a$ is related to $b$ by a relation $R$ then it is denoted by $aRb$. If $R$ is a relation defined from $A$ to $A$ then we say that $R$ is a relation defined on $A$.

Thus, if $R$ is a relation defined on $A$ then $R$ is a subset of $A \times A$.

**Examples**

1. If $A = \{a, b, c\}$ and $B = \{p, q\}$ then $R = \{(a, q), (b, p), (c, p)\}$ is a relation. But $R_1 = \{(a, p), (b, q), (c, r)\}$ is not a relation from $A$ to $B$ as $R_1$ is not a subset of $A \times B$.

2. Let $A = \mathbb{Z}, B = \mathbb{Z}$. A relation $R_1$ can be defined as $R_1 = \{(x, y) | y = x + 1\}$.

3. $A = \{a, b, c, d\}, B = A$. Define a relation $R_2$ on $A$ by

$$R_2 = \{(a, b), (b, b), (b, a), (c, b), (c, c), (b, c), (d, a), (d, d), (a, d)\}.$$

**Definition 2.8** A relation $R$ defined on a set $A$ is said to be a reflexive relation if, for every $x \in A$, $(x, x) \in R$.

In other words, $R$ is a reflexive relation on $A$ if every element of $A$ is related to itself, that is, $xRx$ for every $x \in A$.

**Example 2.12** If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3)\}$ is a reflexive relation but $R_1 = \{(1, 1), (2, 2), (1, 3), (2, 3)\}$ is not a reflexive relation as $(3, 3) \notin R_1$.

**Definition 2.9** A relation $R$ defined on a set $A$ is said to be a symmetric relation if $(x, y) \in R$ implies that $(y, x) \in R$.

In other words, $R$ is a symmetric relation on $A$ if $a$ is related to $b$ implies that $b$ is related to $a$.

**Example 2.13** If $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1), (3, 3)\}$ is a symmetric relation but $R_1 = \{(1, 1), (2, 2), (1, 3), (2, 3)\}$ is not a symmetric relation as $(1, 3) \in R_1$ but $(3, 1) \notin R_1$.

**Definition 2.10** A relation $R$ on a set $A$ is called a transitive relation if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

In other words, if $x$ is related to $y$ and $y$ is related to $z$, then $x$ is related to $z$. For example, if $A = \{1, 2, 3\}$ then
$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ is a transitive relation but $R_1 = \{(1, 1), (2, 2), (1, 2), (2, 3)\}$ is not a transitive relation as $(1, 2) \in R_1, (2, 3) \in R_1$ but $(1, 3) \notin R_1$.

**Definition 2.11** A relation $R$ on a set $A$ is said to be an equivalence relation if $R$ is reflexive, symmetric and transitive relation.

In other words, a relation $R$ on a set $A$ is an equivalence relation if

1. For every $a \in A$, $(a, a) \in R$.
2. If $(a, b) \in R$ then $(b, a) \in R$.
3. If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

**Example 2.14**

1. If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3)\}$. Then $R$ is a reflexive, symmetric and transitive relation. Hence $R$ is an equivalence relation. But

$$R_1 = \{(1, 1), (2, 2), (1, 2), (2, 3)\}$$



is not a reflexive relation defined on $A$ as $(3, 3) \notin R_1$ and hence it is not an equivalence relation.

Consider, $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$. $R_2$ is a reflexive relation on $A$ but it is not symmetric. Hence, it is not an equivalence relation.

2. Let $A = \{1, 2, 3, 4, 5\}$. Define a relation $R$ on $A$ by $xRy$ if and only if $x < y$. Note that the relation $R$ is neither reflexive nor symmetric. If $xRy$ and $yRz$ then $x < y$ and $y < z$. This implies that $x < z$, that is, $xRz$. Hence, $R$ is transitive.
3. Let $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and $R$ be a relation defined on $A$ by $aRb$ if and only if $a$ divides $b$. Note that $R$ is reflexive as for every $a \in A, a$ divides itself. Also, $6R30$ as 6 divides 30. But 30 does not divide 6. Hence, 30 is not related to 6. Hence, $R$ is not a symmetric relation. It is easy to see that $R$ is a transitive relation. Hence the relation $R$ is reflexive and transitive but not a symmetric relation.

**Definition 2.12** Let $A$ be a nonempty set and $\sim$ be an equivalence relation defined on $A$. For $a \in A$, the equivalence class of $a$, denoted by $[a]$, is defined as the set given by $\{x \in A | x \sim a\}$.

**Remark 2.5** Equivalence class of $a$ is also denoted by $E(a)$, cl$(a)$ or by $\bar{a}$. We note that for every $a \in A$, $a \in [a]$ as $a \sim a$. Thus, every equivalence class is nonempty. The set of all the equivalence classes is denoted by $A/\sim$ .

**Lemma 2.1** Let $A$ be a nonempty set and $\sim$ be an equivalence relation defined on $A$. Then, any two equivalence classes are either equal or mutually disjoint.
**Proof.** Let $a, b \in A$. We have to prove that $[a] = [b]$ or $[a] \cap [b] = \phi$. Suppose $[a] \cap [b] \neq \phi$. Then, there is an element, say $c$, such that $c \in [a] \cap [b]$. Hence, $c \sim a$ and $c \sim b$. Since, $\sim$ is an equivalence relation, we get by symmetry $a \sim c$. Also, by transitivity, we get $a \sim b$.

Suppose, $x \in [a]$. Hence, $x \sim a$. But, $a \sim b$. Hence, by transitivity, we get $x \sim b$. This implies that $[a] \subset [b]$. Similarly, we can prove that $[b] \subset [a]$. Hence, $[a] = [b]$.

**Definition 2.13** Let $A$ be a nonempty set and $S$ be a family of subsets of $A$. $S$ is said to be a partition of $A$ if the following conditions are satisfied:

1. Every element of $S$ is nonempty.
2. Intersection of two distinct elements of $S$ is an empty set.
3. Union of all the elements of $S$ equals $A$.

In other words, if $S = \{S_k\}$ then $S$ is said to be a partition of $A$ if $S_k \neq \phi$ for every $k$, $S_j \cap S_k = \phi$ whenever $j \neq k$ and $\cup_k S_k = A$.

**Theorem 2.6** Let $A$ be a nonempty set and $\sim$ be an equivalence relation defined on $A$. Then $A/\sim$ forms a partition of $A$.
**Proof.** We note that by Lemma 2.1, any two equivalence classes are either equal or disjoint. Also, for every $a \in A$, $a \in [a]$ as $a \sim a$. Hence, union of all the equivalence classes equals $A$. Hence, $A/\sim$ forms a partition of $A$.

**Example 2.15** Let $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Define a relation $r$ on $S$ as $a\ r\ b$ if and only if the English spellings of $a$ and $b$ begin with the same letter. Thus, 2 is related to 3 as spellings of both 2 and begin with 'T'. Hence,

$$\begin{aligned} r &= \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7), \\ &\quad (2,3),(3,2),(4,5),(5,4),(6,7),(7,6)\}. \end{aligned}$$

Note that $r$ is an equivalence relation. The equivalence classes are given by $[0] = \{0\}, [2] = \{2, 3\}, [6] = \{6, 7\}, [1] = \{1\}$ and $[4] = \{4, 5\}$.

**Example 2.16** Let $R$ be a relation defined on $\mathbb{Z}$ such that $aRb$ if and only if 5 divides $(a-b)$. We note that $aRa$ for every $a \in \mathbb{Z}$ as 5 divides $a-a = 0$. Hence, $R$ is a reflexive relation. Also, if $aRb$ then 5 divides $(a-b)$. Hence, 5 divides $(b-a)$, that is, $bRa$. This shows that $R$ is a symmetric relation. Suppose $aRb$ and $bRc$ then 5 divides $(a-b)$ and also $(b-c)$. Hence, 5 divides $(a-b)+(b-c) = (a-c)$, that is, $aRc$. This implies that $R$ is a transitive relation. Hence, $R$ is an equivalence relation. When we divide any integer by 5, all the possible remainders are $0, 1, 2, 3$ or $4$. Hence, we get five equivalence classes, $[0], [1], [2], [3]$ and $[4]$.



## Exercise 2.2

1. For each of the following relations $R$ defined on $\mathbb{Z}^\star$, determine which of the ordered pairs belong to $R$.
   a) $xRy$ if and only if $x$ divides $y$; $(2,3), (2,4), (2,8), (2,17)$
   b) $xRy$ if and only if $x \leq y$; $(2,3), (3,2), (2,4), (5,8)$
   c) $xRy$ if and only if $y = x^2$; $(1,1), (2,3), (2,4), (2,6)$

2. Determine which of the following are reflexive, symmetric, transitive and equivalence relations on the set $A$:

   (a) Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation $R$ on $A$ by $R = \{(i,j) | \mid i - j \mid = 2\}$.

   (b) $A =$ the set of all the lines in the planes. Define a relation $R$ on $A$ by $lRm$ if and only if $l$ is parallel to $m$.

   (c) $A =$ the set of all the lines in the planes. Define a relation $R$ on $A$ by $lRm$ if and only if $l$ is perpendicular to $m$.

   (d) Suppose $A = \{0, 1, 2, 3\}$ and $R$ is a relation given by
   $$\begin{aligned} R &= \{(0,0),(1,1),(2,2),(3,3),(1,2),(2,1), \\ &\quad (3,2),(2,3),(3,1),(1,3)\} \end{aligned}$$

   (e) $A = \mathbb{R}$. Define a relation $R$ on $A$ by $xRy$ if and only if $\mid x - y \mid \leq 7$.

   (f) Let $R$ be the relation on $\mathbb{Z} \times \mathbb{Z}$ given by $(x,y)R(s,t)$ if and only if $x < s$ and $y < t$.

   (g) Let $R$ be the relation on $\mathbb{Z} \times \mathbb{Z}$ given by $(x,y)R(s,t)$ if and only if $x + t = y + s$.

   (h) Let $R$ be the relation on $\mathbb{Z} \times \mathbb{Z}$ given by $(x,y)R(s,t)$ if and only if $xt = ys$.

   (i) Let $R$ be the relation on $\wp(\mathbb{Z})$ given by $ARB$ if and only if $A \bigcap B \neq \phi$, $A, B \in \wp(\mathbb{Z})$.

   (j) Let $R$ be the relation on the set of integers $\mathbb{Z}$ given by $xRy$ if and only if $x$ and $y$ share a common factor other than $+1$ or $-1$.

(k) Let $R$ be the relation on $\mathbb{Z} \times \mathbb{Z}$ given by $(x, y)R(s, t)$ if and only if $x < s$ or $y < t$.

3. Give one example of a relation for each of the following:
a) reflexive, symmetric but not transitive.
b) symmetric, transitive but not reflexive.
c) reflexive, transitive but not symmetric.
d) reflexive, neither symmetric nor transitive.
e) symmetric, neither reflexive nor transitive.
f) transitive but neither reflexive nor symmetric.
g) reflexive, symmetric and transitive.
h)neither reflexive nor symmetric nor transitive.

## 2.5 Congruences

A *congruence* is a convenient statement about divisibility. It often makes it easier to discover proofs. The notion of congruence was introduced by C. F. Gauss (1777-1855) in his famous book Disquisitiones Arithmeticae, written at age 24. It gained ready acceptance as a fundamental tool for the study of number theory.

**Definition 2.14** Let $m$ be a non-zero integer. The integers $a$ and $b$ are said to be congruent $modulo\ m$ if and only if $m|(a-b)$, and written $a \equiv b \pmod{m}$.

Since, $(a-b)$ is divisible by $m$ if and only if $(a-b)$ is divisible by $-m$, we will confine our attention to a positive modulus.

**Theorem 2.7** Let $a, b, c, d, x, y$ denote integers. Then,

1. $a \equiv b \pmod{m}$, $b \equiv a \pmod{m}$, and $(a-b) \equiv 0 \pmod{m}$ are equivalent statements.

2. If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

3. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then
$$ax + cy \equiv bx + dy \pmod{m}.$$



4. If$a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then$ac \equiv bd \pmod{m}$.

5. If $a \equiv b \pmod{m}$ and $d|m$, then $a \equiv b \pmod{d}$.

**Proof**:

1. Suppose $a \equiv b \pmod{m}$. Then, by definition, $m|(a-b)$. Now, $m|(a-b)$ if and only if $m|(b-a)$ if and only if $m|(a-b)-0$. Hence, $a \equiv b \pmod{m}$, $b \equiv a \pmod{m}$, and $a-b \equiv 0 \pmod{m}$ are equivalent statements.

2. If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $m|(a-b)$ and $m|(a-c)$. Hence, $m|(a-c)$ i.e. $a \equiv c \pmod{m}$.

3. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $m|(a-b)$ and $m|(c-d)$. Hence, $m|(a-b)x$ and $m|(c-d)y$. Hence,
$$m|(ax+cy)-(bx+dy) \text{ i. e. } ax+cy \equiv bx+dy \pmod{m}.$$

4. $m|(a-b)$ and $m|(c-d) \Rightarrow m|[c \cdot (a-b) + b \cdot (c-d)]$
$\Rightarrow m|(ac-bd) \Rightarrow ac \equiv bd \pmod{m}$.

5. If $a \equiv b \pmod{m}$ then $m|a-b$. But $d|m$, hence, $d|a-b$ i.e. $a \equiv b \pmod{m}$.

**Remark 2.6** If we define a relation $R$ on the set of integers by $aRb$ if $a$ is congruent to $b$ modulo $m$, then by the above theorem, it follows that $R$ is an equivalence relation.

**Theorem 2.8** Let $f(x)$ denote a polynomial with integral coefficients. If $a \equiv b \pmod{m}$, then $f(a) \equiv f(b) \pmod{m}$.
**Proof**: Assume that $f(x) = c_0 + c_1x + \cdots + c_nx^n$, where $c_i$'s are integers. Since, $a \equiv b \pmod{m}$, we get $a^2 \equiv b^2 \pmod{m}, \ldots, a^n \equiv b^n \pmod{m}$. Hence, for every $j,\ 0 \leq j \leq n$, we get $c_ja^j \equiv c_jb^j \pmod{m}$. Hence,
$$\sum_{j=0}^{n} c_ja^j \equiv \sum_{j=0}^{n} c_jb^j \pmod{m}, \text{ that is } f(a) \equiv f(b) \pmod{m}.$$

**Theorem 2.9** Let $a, b, x, y, m, m_1, \ldots, m_r$ be integers. Then,

1. $ax \equiv ay \pmod{m}$ if and only if $x \equiv y \left(\bmod\ \dfrac{m}{(a,m)}\right)$.

2. If $ax \equiv ay \pmod{m}$ and $(a,m) = 1$ then $x \equiv y \pmod{m}$.

3. $x \equiv y \pmod{m_i}$ for $i = 1, 2, \ldots, r$ if and only if $x \equiv y \pmod{[m_1, m_2, \ldots, m_r]}$.

**Proof.**

1. If $ax \equiv ay \pmod{m}$ then $ax - ay = mq$ for some integer $q$. Hence, we have $\dfrac{a}{(a,m)}(x-y) = \dfrac{m}{(a,m)}q$ and thus $\dfrac{m}{(a,m)} | \dfrac{a}{(a,m)}(x-y)$. But $\left(\dfrac{a}{(a,m)}, \dfrac{m}{(a,m)}\right) = 1$. Hence, we get $\dfrac{m}{(a,m)} | (x-y)$, that is, $x \equiv y \left(\bmod\ \dfrac{m}{(a,m)}\right)$.

   Conversely, if $x \equiv y \left(\bmod\ \dfrac{m}{(a,m)}\right)$ then $\frac{m}{(a,m)} | (x-y)$, hence, $m | (a,m)(x-y)$. This implies that $m | a(x-y)$, that is,
   $$ax \equiv ay \pmod{m}.$$

2. If $ax \equiv ay \pmod{m}$ and $(a,m) = 1$ then $x \equiv y \left(\bmod\ \frac{m}{a,m}\right)$. But $(a,m) = 1$ hence we get $x \equiv y \pmod{m}$.

3. If $x \equiv y \pmod{m_i}$ for $i = 1, 2, \ldots, r$ then $m_i | (x-y)$ for $i = 1, \ldots, r$. That is, $(x-y)$ is a common multiple of $m_1, \ldots, m_r$ and therefore $x \equiv y$ (mod $[m_1, m_2, \ldots, m_r]$).

   Conversely, if $x \equiv y \pmod{[m_1, m_2, \ldots, m_r]}$ then for $i = 1, \ldots, r$, $m_i | [m_1, \ldots, m_r]$. Hence, $x \equiv y \pmod{m_i}$ for $i = 1, 2, \ldots, r$.

**Proposition 2.1** If $b \equiv c \pmod{m}$ then $(b,m) = (c,m)$.
**Proof.** Since, $b \equiv c \pmod{m}$ we get $b = c + qm$. Hence, $(c,m) | b$ and hence, $(c,m) | (b,m)$. Also, $c = b - qm$ implies that $(b,m) | c$ and hence $(b,m) | (c,m)$. As both $(b,m)$ and $(c,m)$ are positive, we get $(b,m) = (c,m)$.



**Example 2.17** Find the remainder when $13^{73} + 14^3$ is divided by 11.
**Solution.** We note that $13 \equiv 2 \pmod{11}$ and $14 \equiv 3 \pmod{11}$. Hence, $14^3 \equiv 3^3 \pmod{11}$, that is,
$$14^3 \equiv 5 \pmod{11}. \qquad (1)$$
Also, $2^5 = 32 \equiv -1 \pmod{11}$. Hence, $2^{70} \equiv 1 \pmod{11}$. Since $2^3 \equiv 8 \pmod{11}$, we get $2^{73} \equiv 8 \pmod{11}$. Thus,
$$13^{73} \equiv 8 \pmod{11}. \qquad (2)$$
Adding the congruences (1) and (2), we get
$$13^{73} + 14^3 \equiv 8 + 5 \equiv 2 \pmod{11}.$$
Hence, 2 is the remainder when $13^{73} + 14^3$ is divided by 11.

**Example 2.18** Show that a number is divisible by 3 if and only if the sum of its digits is divisible by 3.
**Solution.** Let $n$ be a given number. $n$ can be written as
$$n = n_0 + 10n_1 + \cdots + 10^k n_k,$$
where $0 \le n_0, n_1, \ldots, n_k \le 9$. Note that $10 \equiv 1 \pmod{3}$. Hence, for every positive integer $m$ $10^m \equiv 1 \pmod{3}$. Hence, $n \equiv n_0 + n_1 + \cdots + n_k \pmod{3}$. This implies that $n$ is divisible by 3 if and only if the sum of its digits is divisible by 3.

**Example 2.19** If $p$ and $q$ are primes such that $p = q+2$, prove that $p^p + q^q$ is a multiple of $p+q$.
**Solution.** We note that as $p$ and $q$ are primes such that $p = q + 2$, both $p$ and $q$ are odd primes. Hence, $q - 1$ is even. Consider
$$\begin{aligned} p^p + q^q &= (p+q-q)^p + q^q \equiv (-q)^p + q^q \pmod{p+q} \\ &\equiv -q^p + q^q \equiv q^q(1-q^2) \pmod{p+q} \end{aligned}$$
Now $p+q = 2q+2$ and $2 | q-1$. Hence, $p+q = 2(q+1)$ divides $1-q^2$. Hence, $p^p + q^q \equiv 0 \pmod{p+q}$, that is, $p^p + q^q$ is a multiple of $p+q$.

**Example 2.20** If $a, b$ are integers, $p$ a prime, then show that

$$(a+b)^p \equiv a^p + b^p \pmod p.$$

**Solution.** We note that

$$\begin{aligned}(a+b)^p &= a^p + \binom{p}{1} a^{p-1}b + \cdots + \binom{p}{p-1} ab^{p-1}b + b^p \\ &\equiv a^p + b^p \pmod p\end{aligned}$$

as $\binom{p}{1}, \ldots, \binom{p}{p-1}$ are divisible by $p$.

## 2.6 Residue Classes

Let $m$ be a positive integer and $a, b \in \mathbb{Z}$. Define $a \sim b$ if and only if $a \equiv b \pmod m$. The first part of the Theorem 2.7 shows that this is a symmetric relation, while the second part of Theorem 2.7 tells us that it is a transitive relation. Since, every non-zero integer divides $0$, we get that $a \equiv a \pmod m$, that is, $a \sim a$ for all $a \in \mathbb{Z}$. Hence, this relation is a reflexive relation. Since, this relation is reflexive symmetric and transitive relation, this is an equivalence relation.

If $m = 2$ then the equivalence relation gives us equivalence classes with which we are familiar. The class of 0 is the set of all even numbers and the equivalence class of 1 is the set of all odd numbers. More generally, the equivalence class of $a$ consists of all $a + mk$, where $k$ ranges over the set of all integers. We denote the equivalence class of $a$ either by $[a]$ or $\bar{a}$. and we call it congruence class of $a$ or residue class of $a$. $a$ is called as a representative of $[a]$. $[a]$ can also be represented by $a + mk$. Observe that if $[b] = [c]$ then $b = mk + c$ for some integer $k$. Hence, $m | b - c$. When $m = 2$, we have exactly two equivalence classes, $[0]$ and $[1]$. The $[0]$ is called as the set of all even numbers and $[1]$ is called as the set of all odd integers.

Given any integer $b$, by division algorithm, there exists unique integers $q$ and $r$ such that $b = mq + r$, where $0 \leq r < m$, thus $[b] = [r]$. So the $n$ classes $[0], [1], \ldots, [m-1]$ give us all the congruence classes. Observe that all these classes are distinct. For, if $[b] = [c]$ where $0 \leq c < b < m$



then $m | b - c$. But $0 < b - c < m$, a contradiction. (We have proved that if $a | b, b \neq 0$ then $|a| \leq |b|$.)

We shall now introduce two operations $+$ and $\cdot$, in $\mathbb{Z}_n$. We define $[a] + [b] = [a + b]$. We must prove that this is a well defined operation, i.e., we must prove that that the addition is independent of the representative of the equivalence class chosen. In other words, if $[a] = [a']$ and $[b] = [b']$ then we must prove that $[a + b] = [a' + b']$. If $[a] = [a']$ and $[b] = [b']$, then $a \equiv a' \pmod m$ and $b \equiv b' \pmod m$. Hence, $a + b \equiv a' + b' \pmod m$. But this means that $[a+b] = [a'+b']$. Hence, $+$ is a well defined operation. Similarly, we define $\cdot$ as $[a][b] = [ab]$. This is also a well defined operation. We leave it as an exercise to the reader as it is exactly analogous.

Note that

$$\begin{aligned}[a] + [b] &= [a + b] = [b + a] = [b] + [a] \\ \text{and } [a] \cdot [b] &= [ab] = [ba] = [b] \cdot [a].\end{aligned}$$

In other words, both these operations are commutative. Also,

$$\begin{aligned}([a] + [b]) + [c] &= [a + b] + [c] = [(a + b) + c] \\ &= [a + (b + c)] = [a] + ([b] + [c]).\end{aligned}$$

Similarly, we can show that $([a] \cdot [b]) \cdot [c] = [a] \cdot ([b] \cdot [c])$. Also,

$$[a] + [0] = [a], [a] + [-a] = [0], \text{ and } [a] \cdot [1] = [a].$$

The addition and multiplication of residue classes can also be shown by so called multiplication tables. While illustrating the multiplication tables, we use $\bar{a}$ instead of $[a]$. For example, the addition and multiplication tables of residue classes modulo 6 are given below.

| $+$ | $\bar{0}$ | $\bar{1}$ | $\bar{2}$ | $\bar{3}$ | $\bar{4}$ | $\bar{5}$ |
|---|---|---|---|---|---|---|
| $\bar{0}$ | $\bar{0}$ | $\bar{1}$ | $\bar{2}$ | $\bar{3}$ | $\bar{4}$ | $\bar{5}$ |
| $\bar{1}$ | $\bar{1}$ | $\bar{2}$ | $\bar{3}$ | $\bar{4}$ | $\bar{5}$ | $\bar{0}$ |
| $\bar{2}$ | $\bar{2}$ | $\bar{3}$ | $\bar{4}$ | $\bar{5}$ | $\bar{0}$ | $\bar{1}$ |
| $\bar{3}$ | $\bar{3}$ | $\bar{4}$ | $\bar{5}$ | $\bar{0}$ | $\bar{1}$ | $\bar{2}$ |
| $\bar{4}$ | $\bar{4}$ | $\bar{5}$ | $\bar{0}$ | $\bar{1}$ | $\bar{2}$ | $\bar{3}$ |
| $\bar{5}$ | $\bar{5}$ | $\bar{0}$ | $\bar{1}$ | $\bar{2}$ | $\bar{3}$ | $\bar{4}$ |

**Addition table for $\mathbb{Z}_6$**

| $\cdot$ | $\bar{0}$ | $\bar{1}$ | $\bar{2}$ | $\bar{3}$ | $\bar{4}$ | $\bar{5}$ |
|---|---|---|---|---|---|---|
| $\bar{0}$ | $\bar{0}$ | $\bar{0}$ | $\bar{0}$ | $\bar{0}$ | $\bar{0}$ | $\bar{0}$ |
| $\bar{1}$ | $\bar{0}$ | $\bar{1}$ | $\bar{2}$ | $\bar{3}$ | $\bar{4}$ | $\bar{5}$ |
| $\bar{2}$ | $\bar{0}$ | $\bar{2}$ | $\bar{4}$ | $\bar{0}$ | $\bar{2}$ | $\bar{4}$ |
| $\bar{3}$ | $\bar{0}$ | $\bar{3}$ | $\bar{0}$ | $\bar{3}$ | $\bar{0}$ | $\bar{3}$ |
| $\bar{4}$ | $\bar{0}$ | $\bar{4}$ | $\bar{2}$ | $\bar{0}$ | $\bar{4}$ | $\bar{2}$ |
| $\bar{5}$ | $\bar{0}$ | $\bar{5}$ | $\bar{4}$ | $\bar{3}$ | $\bar{2}$ | $\bar{1}$ |

**Multiplication Table for $\mathbb{Z}_6$**

Observe that $\bar{a}\bar{1} = \bar{a}$ for all the elements in $\mathbb{Z}_6$. Also, $\bar{5}\bar{5} = \bar{1}$. But there does not exist any $\bar{a} \in \mathbb{Z}_6$ such that $\bar{2}\bar{a} = \bar{1}$.

We also note that if $[a] = [b]$ then $(a, m) = (b, m)$. For if $[a] = [b]$ then $b = mq + a$ for some integer $q$. Hence,if $(a, m) = d$ then $d|a, d|m$ and also $d|b$, i.e., $(a, m)|(b, m)$. Similarly, $(b, m)|(a, m)$. Since, both $(a, m)$ and $(b, m)$ are positive, they are equal to each other. Hence, if $(a, m) = 1$ and $b \in [a]$ then $b$ and $m$ are relatively prime to each other. In this case, every element of $[a]$ is relatively prime to $m$.

**Definition 2.15** Let $[a] \in \mathbb{Z}_m$ be an equivalence class of $a$. $[a]$ is called a prime residue class modulo $m$ if $(a, m) = 1$. The set of all prime residue classes modulo $m$ is denoted by $\mathbb{Z}_m^\star$.

Thus, $\mathbb{Z}_m^\star = \{[a] | (a, m) = 1\}$. This set is also denoted by $U_m$. A prime residue class is also called as a reduced residue class.

We now give the multiplication table for $\mathbb{Z}_6^\star$ and for $\mathbb{Z}_8^\star$. Recall that $\mathbb{Z}_6^\star = \{\bar{1}, \bar{5}\}$ and $\mathbb{Z}_6^\star = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$.

| $\cdot$ | $\bar{1}$ | $\bar{5}$ |
|---|---|---|
| $\bar{1}$ | $\bar{1}$ | $\bar{5}$ |
| $\bar{5}$ | $\bar{5}$ | $\bar{1}$ |

**Multiplication Table for $\mathbb{Z}_6^\star$**

| $\cdot$ | $\bar{1}$ | $\bar{3}$ | $\bar{5}$ | $\bar{7}$ |
|---|---|---|---|---|
| $\bar{1}$ | $\bar{1}$ | $\bar{3}$ | $\bar{5}$ | $\bar{7}$ |
| $\bar{3}$ | $\bar{3}$ | $\bar{1}$ | $\bar{7}$ | $\bar{5}$ |
| $\bar{5}$ | $\bar{5}$ | $\bar{7}$ | $\bar{1}$ | $\bar{3}$ |
| $\bar{7}$ | $\bar{7}$ | $\bar{5}$ | $\bar{3}$ | $\bar{1}$ |

**Multiplication Table for $\mathbb{Z}_8^\star$**

**Remark 2.7** The Euler's $\phi$ function is defined as the number of positive integers less than or equal to $m$ and relatively prime to $m$. Thus, $\phi(1) = 1$, $\phi(2) = 1$. $\phi(8) = 4$ as $1, 3, 5, 7$ are the exactly four integers less than or equal to $8$ and relatively prime to $8$. If $p$ is a prime, then $\phi(p) = p - 1$ and it can be easily seen that $\phi(p^n) = p^n - p^{n-1}$. Euler's $\phi$ function has many interesting properties. For example, if $(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$.

**Remark 2.8 (Euler's theorem)** Let $a, m$ be relatively prime integers (i.e. $(a, m) = 1$). Then $a^{\phi(m)} \equiv 1 \pmod{m}$.



Following is an easy consequence of Euler's theorem.

**Remark 2.9 (Fermat's theorem)** Let $p$ be a prime and $a$ be an integer. Then $a^p \equiv a \pmod{p}$.

## Exercise Set - 2.3

1. Verify that $([a] \cdot [b]) \cdot [c] = [a] \cdot ([b] \cdot [c])$. Also, $[a] + [0] = [a]$, $[a] + [-a] = [0]$, and $[a] \cdot [1] = [a]$.
2. Show that a number is divisible by $9$ if and only if the sum of its digits is divisible by $9$.
3. Show that a number is divisible by $4$ if and only if the number formed by its tens digit and its units digit is divisible by $4$.
4. Show that $(a+1)^p - a^p - 1$ is divisible by $p$. Also, show that if $p$ divides $a^p - a$ then $p$ divides $(a+1)^p - a - 1$.
5. Prove that for every positive integer $n$,
   (a) $1^n + 8^n - 3^n - 6^n$ is divisible by $10$.
   (b) $2903^n - 803^n - 464^n + 261^n$ is divisible by $1897$.
6. Prove that the expressions $2x + 3y$ and $9x + 5y$ are divisible by $17$ for the same set of integral values of $x$ and $y$.
7. Find the remainder when $4^{37} + 82$ is divided by $7$.

# Chapter 3

# Polynomials

## 3.1 Introduction

In this chapter we study $\mathbb{R}[x]$, the set of polynomials in one variable $x$ with real coefficients. We define equality of polynomials, zero polynomial, degree of a nonzero polynomial. Then, in $\mathbb{R}[x]$ we define addition and multiplication of polynomials and some properties related to these operations. We shall further discuss Remainder theorem and Factor theorem using Division Algorithm theorem.

## 3.2 Definitions and Remarks:

**Definition 3.1 (Polynomial)** An expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_i$'s are real numbers and $n$ is a non-negative integer is called a polynomial over $\mathbb{R}$ in $x$ and is denoted by $p(x)$. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $a_i$'s are called the coefficients of polynomial $p(x)$. The set of all polynomials in one variable $x$ with real coefficients is denoted by $\mathbb{R}[x]$. Similarly, the set of all polynomials in one variable $x$ with complex coefficients is denoted by $\mathbb{C}[x]$.

**Definition 3.2 (Equality of polynomials)** Let $p(x)$ and $q(x)$ be polynomials where $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in \mathbb{R}$ and $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, $b_i \in \mathbb{R}$ are said to be equal if $n = m$ and $a_i = b_i$, for $i = 0, 1, \cdots, n$

**Definition 3.3 (Zero Polynomial)** If all the coefficients of a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ are zero then it is called a zero polynomial i.e. $a_i = 0$, for $i = 0, 1, \cdots, n$.





**Definition 3.4 (Leading Coefficient)**
Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial where $a_n \neq 0$. Then $a_n$, the coefficient of $x^n$ is called its leading coefficient and $n$ is called the degree of the non-zero polynomial, denoted by $\deg p(x)$..

**Definition 3.5 (Monic Polynomial)** A polynomial with leading coefficient one is called a monic polynomial.

**Some remarks and examples** :

1. A nonzero polynomial of zero degree is called a **constant polynomial**.
2. A polynomial of degree one is called a **linear polynomial**.
3. A polynomial of degree two is called a **quadratic polynomial**.
4. A zero polynomial has no degree (Convention henceforth).
5. $f(x) = 1 + x - x^2 + x^3$ is a monic polynomial of degree 3 in $\mathbb{R}[x]$ since the coefficient of $x^3$ is 1.
6. $p(x) = 3x - 2$ is a linear polynomial in $\mathbb{R}[x]$ and $q(x) = 7$ is a constant polynomial in $\mathbb{R}[x]$.
7. $g(x) = 0$ is the zero polynomial in $\mathbb{R}[x]$.

## 3.3 Addition and Multiplication

**Definition 3.6** Let $p(x)$ and $q(x)$ be polynomials, where

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ q(x) &= b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0. \end{aligned}$$

Then, the addition of two polynomials is defined as

$$p(x) + q(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$$

where $c_i = a_i + b_i$, for $i = 0, 1, 2, \cdots, k$ where $k = \max\{m, n\}$. and the multiplication is defined as

$$p(x)q(x) = c_{m+n}x^{m+n} + \cdots c_k x^k + \cdots + c_1 x + c_0$$

where $c_k = a_k b_0 + a_{k-1}b_1 + \cdots + a_1 b_{k-1} + a_0 b_k$.

**Example 3.1** Let $p(x) = x^2 - x - 2$ and $q(x) = 5x^3 - x + 3$.
(a) Find $p(x) + q(x)$, (b) Find $p(x)q(x)$,
(c) What is the degree of $p(x)q(x)$ ?
(d) Is the polynomial $q(x)$ monic? If not, what is the leading coefficient of $q(x)$?
Solution : (a)

$$\begin{aligned} p(x) + q(x) &= (x^2 - x - 2) + (5x^3 - x + 3) \\ &= 5x^3 + x^2 - x - x - 2 + 3 = 5x^3 + x^2 - 2x + 1. \end{aligned}$$

(b) $p(x)q(x) = (x^2 - x - 2)(5x^3 - x + 3) = x^2(5x^3 - x + 3) - x(5x^3 - x + 3) - 2(5x^3 - x + 3) = 5x^5 - x^3 + 3x^2 - 5x^4 + x^2 - 3x - 10x^3 + 2x - 6 = 5x^5 - 5x^4 - 11x^3 + 4x^2 - x - 6$. (c) $\deg(q(x)) = 3$.
(d) Polynomial $q(x)$ is not a monic polynomial since the leading coefficient is 5.

**Example 3.2** Let $p(x), q(x) \in \mathbb{R}[x]$. Show that

$$\deg[p(x) + q(x)] \leq \max\{\deg p(x), \deg q(x)\}\,.$$

Solution : Let $\deg p(x) = m$ and $\deg q(x) = n$. If $p(x) + q(x)$ is a non zero polynomial then there exist largest $k$ such that $a_k + b_k \neq 0$. We show that $k \leq \max\{m, n\}$.

Suppose $k > \max\{m, n\} \Rightarrow k > m$ and $k > n \Rightarrow b_k = 0$ and $a_k = 0 \Rightarrow a_k + b_k = 0$, which is a contradiction. Hence $k \leq \max\{m, n\}$.

**Example 3.3** Show that $\deg p(x)q(x) = \deg p(x) + \deg q(x)$.
Solution : Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and
$q(x) = b_m x^m + \cdots + b_1 x + b_0$. Then



$p(x)q(x) = c_{m+n}x^{m+n} + \cdots + c_1 x + c_0$, where
$c_k = a_0 b_k + a_1 b_{k-1} + \cdots a_k b_0$. Since $a_n \neq 0$ and $b_m \neq 0 \Rightarrow c_{m+n} = a_n b_m \neq 0$.

We show that $m + n$ is the highest power of $x$ in the expansion of $p(x)q(x)$. In other words if $k > m+n$, we must show that $c_k = 0$. Clearly $a_k = 0$ if $k > n$ and $b_k = 0$ if $k > m$.
Thus $c_k = a_0 b_k + a_1 b_{k-1} + \cdots a_{k-m} b_m + a_{k-m+1} b_{m-1} + \cdots a_k b_0 = 0$ (Since $b_k = b_{k-1} = \cdots = b_{m+1} = 0$ and $a_{k-m} = a_{k-m+1} = \cdots = a_k = 0$ if $k > m+n$). Thus the highest power of $x$ in $p(x)q(x)$ is $m+n$. Hence $\deg p(x)q(x) = \deg p(x) + \deg q(x)$.

## 3.4 Divisibility in $\mathbb{R}[x]$

**Definition 3.7** Let $p(x) \in \mathbb{R}[x]$. A nonzero polynomial $q(x) \in \mathbb{R}[x]$ is said to be a divisor of $p(x)$ if there exists $h(x) \in \mathbb{R}[x]$ such that $p(x) = q(x)h(x)$ and write it as $q(x)|p(x)$.

If $p(x) \neq 0$ then $p(x) = 1 \times p(x) \Rightarrow p(x)|p(x)$. Further, if $k \neq 0$ then $p(x) = k\frac{1}{k}p(x) \Rightarrow k|p(x)$.

**Example 3.4** Let $p(x)$ and $q(x)$ be nonzero polynomials in $\mathbb{R}[x]$ such that $p(x)|q(x)$ and $q(x)|p(x)$. Show that there exists $c \neq 0$ such that $p(x) = cq(x)$.
Solution : $p(x)|q(x)$ and $q(x)|p(x) \Rightarrow \exists k_1(x) \neq 0$, $k_2(x) \neq 0$ such that $q(x) = k_1(x)p(x)$ and $p(x) = k_2(x)q(x) \Rightarrow q(x) = k_1(x)k_2(x)g(x) \Rightarrow k_1(x)k_2(x) = 1 \Rightarrow \deg[k_1(x)k_2(x)] = 0$
$\Rightarrow \deg k_1(x) + \deg k_2(x) = 0 \Rightarrow \deg k_1(x) = 0$ and $\deg k_2(x) = 0$. Hence, $k_2(x)$ is a constant polynomial say $k_2(x) = c$. Hence $p(x) = k_2(x)q(x) = cq(x)$.

**Example 3.5** If $p(x)$ and $q(x)$ are nonzero polynomials in $\mathbb{R}[x]$ such that $p(x)|q(x)$ then show that $\deg p(x) \leq \deg q(x)$.
Solution : $p(x)|q(x) \Rightarrow \exists h(x) \in \mathbb{R}[x]$ such that $q(x) = p(x)h(x)$.
Now $\deg g(x) = \deg[p(x)h(x)] = \deg p(x) + \deg h(x) \geq \deg p(x)$
$\Rightarrow \deg p(x) \leq \deg q(x)$, since $\deg h(x) \geq 0$.

**Example 3.6** If $p(x), q(x), r(x) \in \mathbb{R}[x]$ with $p(x) \neq 0$. If $p(x)|q(x)$ and $p(x)|r(x)$ then for $m(x), n(x) \in \mathbb{R}[x]$, show that

$$p(x)|m(x)q(x) + n(x)r(x).$$

Solution: We are given that $p(x)|q(x)$ and $p(x)|r(x)$. Hence there exist polynomials $k_1(x), k_2(x)$ such that $q(x) = p(x)k_1(x), r(x) = p(x)k_2(x)$. Hence,
$m(x)q(x) + n(x)r(x) = f(x)(m(x)k_1(x) + n(x)k_2(x)) = f(x)k(x)$, where $k(x) = m(x)k_1(x) + n(x)k_2(x)$. Thus

$$p(x)|m(x)q(x) + n(x)r(x).$$

**Theorem 3.1 (Division Algorithm)** Let $f(x), g(x) \in \mathbb{R}[x]$ such that $g(x)$ is a non-zero polynomial. There exist unique polynomials $q(x), r(x) \in \mathbb{R}[x]$ such that $f(x) = g(x)q(x) + r(x)$ where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

**Example 3.7** Let $f(x) = 3x^5 - x^3 + 3x - 5$ and $g(x) = x^2 + 7$. Find $q(x), r(x) \in \mathbb{R}[x]$ such that $f(x) = g(x)q(x) + r(x)$.
Solution : By using long division we get, $q(x) = 3x^3 - 22x$ and $r(x) = 157x - 5$. Hence

$$3x^5 - x^3 + 3x - 5 = (x^2 + 7)(3x^3 - 22x) + (157x - 5).$$

**Example 3.8** Let $f(x) = x^3 + x + 3$ and $g(x) = x + 3$. Find $q(x), r(x) \in \mathbb{R}[x]$ such that $f(x) = g(x)q(x) + r(x)$.
Solution : By using long division we get, $q(x) = x^2 - 2$ and $r(x) = 9$. Hence $x^3 + x + 3 = (x + 3)(x^2 - 2) + 9$.

**Definition 3.8 (Greatest Common divisor)** Let $p(x), q(x)$ be nonzero polynomials in $\mathbb{R}[x]$. A polynomial $d(x) \in \mathbb{R}[x]$ is said to be a greatest common divisor (g.c.d.) of $p(x)$ and $q(x)$ if

1. $d(x)|p(x)$ and $d(x)|q(x)$
2. If $c(x)|p(x)$ and $c(x)|q(x)$ then $c(x)|d(x)$



Greatest Common divisor of two polynomials $p(x)$ and $q(x)$ is denoted by

$$(f(x), g(x)) \text{ i.e. } d(x) = (f(x), g(x)).$$

**Remarks:**

1. When we want to calculate a greatest common divisor ($\gcd$) of two nonzero polynomials, we note that it is not unique.

   If $d(x) = (f(x), g(x))$ then for $\alpha \neq 0, \alpha \in \mathbb{R}$, $\alpha d(x)$ is also a $\gcd$ of the given two nonzero polynomials. However, we can make $\gcd$ unique by choosing $\gcd$ to be a monic polynomial.

2. If $f(x), g(x)$ are two polynomials such that $f(x)|g(x)$ and $g(x)|f(x)$ then $f(x) = \alpha g(x)$, where $\alpha \neq 0$. Then, we say that $f(x)$ and $g(x)$ are associates of each other.

3. If the $\gcd$ of two nonzero polynomials is a nonzero constant real number $k$ (say) then by above remark the $\gcd$ is 1. (Choose $\alpha = \frac{1}{k}$ in this case).

**Definition 3.9 (Relatively prime polynomials)** Let $p(x), q(x)$ be two non-zero polynomials in $\mathbb{R}[x]$. $p(x)$ and $q(x)$ are said to be relatively prime if $(p(x), q(x)) = 1$.

**Theorem 3.2 (Existence of gcd)** If $p(x)$, $q(x)$ are two nonzero polynomials in $\mathbb{R}[x]$ then there exists a unique polynomial $d(x)$ which is g.c.d. of $p(x)$ and $q(x)$ and there exist polynomials $m(x), n(x) \in \mathbb{R}[x]$ such that $d(x) = m(x)p(x) + n(x)q(x)$.

**Theorem 3.3 (Euclidean Algorithm to find** $\gcd$ **)**
The procedure to compute the $\gcd$ of two nonzero polynomials in $\mathbb{R}[x]$ using division algorithm is called **Euclidean Algorithm**. The Algorithm is as below :

1. Let $f(x), g(x)$ be nonzero polynomials in $\mathbb{R}[x]$. By Division Algorithm $\exists\ q(x), r(x) \in \mathbb{R}[x]$ such that $f(x) = g(x)q(x) + r(x)$ where $r(x) = 0$ or $\deg\ r(x) < \deg\ g(x)$.

2. If $r(x) = 0$ then the $\gcd d(x) = (f(x), g(x)) = r(x)$. If $r(x) \neq 0$ then again apply Division Algorithm to $g(x)$ and $r(x)$ to get $q_1(x)$ and $r_1(x)$ such that $g(x) = r(x)q_1(x) + r_1(x)$ and further $d(x) = (f(x), g(x)) = (g(x), r(x))$ as in $\mathbb{Z}$.

3. Now, if $r_1(x) \neq 0$ then apply the previous step and continue this process till we get zero remainder. This process stops because at every step the degree of remainder decreases at least by one. The last nonzero remainder is the g.c.d. $d(x)$ of $f(x)$ and $g(x)$.

**Example 3.9** Find the greatest common divisor of $f(x) = x^2 + x + 1$ and $g(x) = x + 1$.
Solution : Dividing f(x) by g(x) we get $x^2 + x + 1 = (x)(x + 1) + 1$ i.e. $f(x) = xg(x) + r(x)$ where $r(x) = 1$.

Again dividing $g(x)$ by $r(x)$ we get $x + 1 = 1(x) + \mathbf{1}$ and again $1 = 1(1) + 0$. Thus $d(x) = (f(x), g(x)) = 1$ which implies that $f(x)$ and $g(x)$ are relatively prime.

**Example 3.10** Find the greatest common divisor of $f(x) = x^4 + 3x^2 + 2$ and $g(x) = x^3 - x^2 + x - 1 \in \mathbb{R}[x]$.
Solution : Dividing $f(x)$ by $g(x)$ we get that,

$f(x) = (x+1)g(x)+(3x^2+3) = q(x)g(x)+r(x)$ where $q(x) = x+1$ and $r(x) = 3x^2 + 3$.
In order to avoid fractions, we divide $3g(x)$ by $r(x) = 3x^2 + 3$ to get $3g(x) = (x - 1)r(x) + 0$.

So the last nonzero remainder is $r(x) = 3x^2 + 3$. Thus the g.c.d. of $f(x)$ and $g(x)$ is $r(x) = 3x^2 + 3$.

Using the fact that g.c.d. of two polynomials is unique upto associates, we conclude that $d(x) = (f(x), g(x)) = x^2 + 1$.

**Example 3.11** Show that $f(x) = x^3 - 2x^2 + 3x - 7$ and $g(x) = x^2 + 2$ are relatively prime.
Solution : Dividing $f(x)$ by $g(x)$ we get that,

$$f(x) = (x - 2)g(x) + (x - 3) = q(x)g(x) + r(x) \text{ (say).}$$



Dividing $g(x)$ by $r(x)$ we get,

$$g(x) = x^2 + 2 = (x + 3)(x - 3) + 11 = q_1(x)r(x) + r_1(x) \text{ (say).}$$

Since the remainder is a nonzero constant it eventually turns out that $\gcd = 11$. Using the fact that $\gcd$ of two polynomials is unique up to associates, we conclude that $(f(x), g(x)) = 1$.

**Theorem 3.4 (Remainder Theorem)** If $f(x) \in \mathbb{R}[x]$ is divided by $(x-\alpha)$ then the remainder is $f(\alpha)$.
Proof : By Division Algorithm there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = (x - \alpha)q(x) + r(x)$ where $r(x) = 0$ or $\deg\ r(x) < \deg\ (x - a) = 1 \Rightarrow \deg\ r(x) = 0$. Hence $r(x)$ is a constant polynomial say $r(x) = c$. Thus $f(x) = (x - \alpha)q(x) + c$. Now put $x = \alpha$ to get $f(\alpha) = (\alpha - \alpha)q(\alpha) + r \Rightarrow f(\alpha) = r$. Thus the remainder is $r = f(\alpha)$.

**Example 3.12** Use Remainder Theorem to compute the remainder when $f(x) = x^4 - 3x^3 - 7x^2 - 2$ is divided by $g(x) = x - 2$.
Solution : Here $g(x) = x - \alpha = x - 2 \Rightarrow \alpha = 2$. By Remainder Theorem $f(\alpha) = f(2) = 2^4 - 3(2)^3 - 7(2)^2 - 2 = -38$ is the remainder.

**Example 3.13** Find the value of $a$, if $x + 2$ is a factor of $x^2 - ax + 6$.
Solution : Divide $f(x) = x^2 - ax + 6$ by $g(x) = x + 2$ by using long division method to get quotient $q(x) = x - (a + 2)$ and remainder $r = 2a - 2$. But $x + 2$ is factor of $x^2 - ax + 6 = 0$, so by Remainder Theorem, $r = 2a - 2 = 0 \Rightarrow a = 1$.

**Theorem 3.5 (Factor Theorem)** $(x - \alpha)$ is a factor of $f(x) \in \mathbb{R}[x]$ if and only if $f(\alpha) = 0$.
Proof : Suppose $f(\alpha) = 0$. By Remainder theorem

$$r(x) = f(\alpha) = 0 \Rightarrow f(x) = (x - \alpha)q(x) \Rightarrow (x - \alpha)$$

is a factor of $f(x)$.

Conversely suppose that $(x - \alpha)$ is a factor of $f(x)$. Hence, there exists $g(x) \in \mathbb{R}[x]$ such that $f(x) = (x - \alpha)g(x)$. Thus, $f(\alpha) = (\alpha - \alpha)g(\alpha) = 0$. Hence the proof.

**Example 3.14** Use Factor Theorem to determine whether $g(x) = x + 3$ is a factor of $f(x) = 3x^3 - 4x + 2$ or not.
Answer : $g(x) = x - \alpha = x + 3 \Rightarrow \alpha = -3$. We check whether $f(\alpha) = 0$ or not. $f(\alpha) = f(-3) = 3(-3)^3 - 4(-3) + 2 = -31 + 14 = -67$. Thus $f(\alpha) = -67 \neq 0$, so by factor theorem $(x - \alpha) = (x + 3)$ is not a factor of $f(x) = 3x^3 - 4x + 2$.

## 3.5 Roots of a Polynomial

We now obtain the relation between the roots and the coefficients of the polynomials.
Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{R}[x]$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of $f(x) = 0$. Then using factor theorem, we get

$$\begin{aligned} & x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = (x - \alpha_1) \cdots (x - \alpha_n) \\ = \ & x^n - (\sum_i \alpha_i)x^{n-1} + (\sum \alpha_i\alpha_j)x^{n-2} \cdots + (-1)^n(\alpha_1\alpha_2..\alpha_n). \end{aligned}$$

Now comparing the coefficients we get the following :

1. $\sum \alpha_i = -a_{n-1} =$ i.e. sum of the roots.
2. $\sum_{i<j} \alpha_i\alpha_j = a_{n-2}$ i.e. sum of the products of roots taken two at a time.
3. Continuing this way, we get $(-1)^n\alpha_1\alpha_2...\alpha_n = a_0$ i.e. product of the roots.

Particular Cases :

1. If $f(x) = x^2 + bx + c = 0$ has 2 roots $\alpha_1$, $\alpha_2$ (say).

   Then we get $f(x) = x^2 + bx + c = (x - \alpha_1)(x - \alpha_2)$

   $\Rightarrow x^2 + bx + c = [x^2 - (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2]$.

   So we get $\alpha_1 + \alpha_2 = -b$ and $\alpha_1\alpha_2 = c$.



2. If $f(x) = x^3 + bx^2 + cx + d = 0$ has 3 roots $\alpha_1, \alpha_2, \alpha_3$ (say) then we get $f(x) = x^3 + bx^2 + cx + d = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$

   $= x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)x - \alpha_1\alpha_2\alpha_3$.

   So comparing we get $\alpha_1 + \alpha_2 + \alpha_3 = -b$, $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = c$ and $\alpha_1\alpha_2\alpha_3 = -d$.

**Example 3.15** Solve the equation $f(x) = 24x^3 - 14x^2 - 63x + 45 = 0$ where one of the roots is double the other.
Solution : Let $\alpha, 2\alpha, \beta$ be the roots of $f(x) = 0$. Then
$\alpha + 2\alpha + \beta = \frac{14}{24}, \alpha\beta + \alpha(2\alpha) + \beta(2\alpha) = -\frac{63}{24}, (\alpha)(2\alpha)(\beta) = -\frac{45}{24}$. Thus $3\alpha + \beta = \frac{7}{12}$, $2\alpha^2 + 3\alpha\beta = -\frac{21}{8}$, $2\alpha^2\beta = -\frac{15}{8}$.
Substitute $\beta = \frac{7}{12} - 3\alpha$ to get $2\alpha^2 + 3\alpha\left(\frac{7}{12} - 3\alpha\right) = -\frac{21}{8}$.
Simplifying we get $8\alpha^2 - 2\alpha - 3 = 0 \Rightarrow \alpha = \frac{3}{4}$ or $-\frac{1}{2}$.
If $\alpha = -\frac{1}{2}$ then $\beta = \frac{24}{12}$, then $2\alpha^2\beta \neq -\frac{15}{8}$.
Thus $\alpha = \frac{3}{4}$ and $\beta = -\frac{5}{3}$ and hence the third root is $2\alpha = \frac{3}{2}$.
Hence the roots are $\frac{3}{4}, \frac{3}{2}, -\frac{5}{4}$.

**Example 3.16** Solve $4x^3 + 20x^2 - 23x + 6 = 0$, two of its roots being equal.
Solution : Let the roots be $\alpha, \alpha, \beta$. Then,

$$\alpha + \alpha + \beta = -\frac{20}{4}, \alpha \cdot \alpha + \beta \cdot \alpha + \alpha \cdot \beta = -\frac{23}{4}, \alpha\alpha\beta = \frac{3}{2}$$

i.e. $2\alpha + \beta = -5$, $\alpha^2 + 2\alpha\beta = -\frac{23}{4}$, $\alpha^2\beta = \frac{3}{2}$.
Since $\beta = -5 - 2\alpha \Rightarrow \alpha^2 + 2\alpha(-5 - 2\alpha) = -\frac{23}{4} \Rightarrow \alpha = \frac{1}{2}, -\frac{23}{6}$.
The third equation $\alpha^2\beta = \alpha^2(-5 - 2\alpha) = \frac{3}{2}$ is satisfied by $\alpha = \frac{1}{2}$ and not by $\alpha = -\frac{23}{6}$. Thus $\alpha = \frac{1}{2}$, $\beta = -5 - 2\alpha = -6$. So the three roots are $\frac{1}{2}, \frac{1}{2}, -6$.

**Example 3.17** The cubic equation $2x^3 - 9x^2 + 12x + \lambda = 0$ has two equal roots. Find $\lambda$ and the roots.
Solution : Let the roots be $\alpha, \alpha, \beta$. Then,
$2\alpha + \beta = \frac{9}{2}$, $\alpha^2 + 2\alpha\beta = 6$, $\alpha^2\beta = -\frac{1}{2}\lambda$.
Since $\beta = \frac{9}{2} - 2\alpha$ the equation $\alpha^2 + 2\alpha\beta = 6$ becomes $\alpha^2 - 3\alpha + 2 = 0 \Rightarrow \alpha = 1, 2$. If $\alpha = 1 \Rightarrow \beta = \frac{5}{2} \Rightarrow \lambda = -5$. If $\alpha = 2 \Rightarrow \beta = \frac{1}{2} \Rightarrow \lambda =$

$-4$. Thus $\lambda = -5$ then the roots are $1, 1, \frac{5}{2}$ and if $\lambda = -4$ then the roots are $2, 2, \frac{1}{2}$.

**Example 3.18** Solve $x^3 - 13x^2 + 15x + 189 = 0$ having given that one root exceeds the other by 2.
Solution : Let the roots be $\alpha, \beta, \gamma$. Then, $\alpha+\beta+\gamma = 13$, $\alpha\beta+\beta\gamma+\gamma\alpha = 15$, $\alpha\beta\gamma = -189$. Suppose $\alpha = \beta + 2 \Rightarrow (\beta + 2) + \beta + \gamma = 14 \Rightarrow \gamma = 11 - 2\beta$. Substitute value of $\gamma$ and $\alpha = \beta + 2$ in the second equation $\alpha\beta + \beta\gamma + \gamma\alpha = 15$ we get $3\beta^2 - 20\beta - 7 = 0 \Rightarrow \beta = 7, -\frac{1}{3}$
$\beta = 7 \Rightarrow \alpha = 9$ and $\gamma = -3$ satisfy the third equation $\alpha\beta\gamma = -189$. But $\beta = -\frac{1}{3} \Rightarrow \alpha = \frac{5}{3}$ leads to $\gamma$ such that $\alpha\beta\gamma \neq -189$. Thus the three roots are $9, 7, -3$.

**Example 3.19** Find the condition that the cubic $x^3 - px^2 + qx - r = 0$ has all roots equal.
Solution : Let the roots be $\alpha, \alpha, \alpha$. Then, $3\alpha = p \Rightarrow \alpha = \frac{p}{3}$, $\alpha^2+\alpha^2+\alpha^2 = q \Rightarrow \alpha^2 = \frac{q}{3}$, $\alpha^3 = r \Rightarrow \frac{p}{3}\frac{q}{3} = \frac{pq}{9} \Rightarrow \alpha\alpha^2 = \frac{pq}{9} \Rightarrow r = \frac{pq}{9}$ is the required condition.

**Example 3.20** Solve $x^3 - 9x^2 + 14x + 24 = 0$, two of its roots being in the ratio 3 : 2.
Solution : Let $3\alpha, 2\alpha, \beta$ be the three roots. Then,
$3\alpha + 2\alpha + \beta = 9 \Rightarrow 5\alpha + \beta = 9 \Rightarrow \beta = 9 - 5a$
$(3\alpha)\beta + (2\alpha)\beta + (3\alpha)(2\alpha) = 14 \Rightarrow 5\alpha\beta + 6\alpha^2 = 14$
$(3\alpha)(2\alpha)(\beta) = -24 \Rightarrow \alpha^2\beta = -4$
Substitute the value of $\beta$ in second equation to get
$5\alpha(9 - 5\alpha) + 6\alpha^2 = 14 \Rightarrow (19\alpha - 7)(\alpha - 2) = 0 \Rightarrow \alpha = 2, \frac{7}{19}$.
If $\alpha = \frac{7}{19} \Rightarrow \beta = \frac{136}{19}$ but then $\alpha^2\beta \neq -4$. Thus $\alpha = 2 \Rightarrow \beta = -1$. Thus the three roots are $3\alpha, 2\alpha, \beta$ i.e. $6, 4, -1$.

**Example 3.21** Solve $x^3 - 6x^2 + 3x + 10 = 0$, the roots being in A.P.
Solution : Let $a - d, a, a + d$ be the three roots. Then,
$a-d+a+a+d = 6 \Rightarrow a = 2$. $(a-d)a+(a-d)(a+d)+a(a+d) = 3$. $\Rightarrow 3a^2 - d^2 = 3 \Rightarrow 12 - d^2 = 3 \Rightarrow d = \pm 3$. Thus the roots $a - d, a, a + d$ are $-1, 2, 5$.



**Example 3.22** Solve $18x^3 + 81x^2 + 121x + 60 = 0$ being given that one root is half the sum of the other two.
Solution : Let the roots be $\alpha, \beta$ and $\gamma$ and $\beta = \frac{\alpha+\gamma}{2}$. Thus the three roots are in A.P and that we can assume that the three roots are $\alpha = a - d, \beta = a, \gamma = a + d$. $a - d + a + a + d = -\frac{9}{2} \Rightarrow a = -\frac{3}{2}$.
$(a - d)a(a + d) = -\frac{60}{18} \Rightarrow d = \pm\frac{1}{6}$. Thus the roots are $-\frac{5}{3}, -\frac{3}{2}, -\frac{4}{3}$.

**Example 3.23** Solve $x^4 + 2x^3 - 12x^2 - 22x + 40 = 0$ whose roots are in A.P.
Solution : Let the roots be $a - 3d, a - d, a + d, a + 3d$. Then, $a - 3d + a - d + a + d + a + 3d = -2 \Rightarrow a = -\frac{1}{2}$.
$(a - 3d)(a - d)(a + d)(a + 3d) = 40 \Rightarrow (4d^2 - 9)(36d^2 + 71) = 0$.
Since $36d^2 + 71 \neq 0$ we have $4d^2 - 9 = 0 \Rightarrow d = \pm\frac{3}{2}$. Thus the four roots are $a - 3d, a - d, a + d, a + 3d$ i.e. $-5, -2, 1, 4$.

**Example 3.24** Solve $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in G.P.
Solution : Let $\frac{a}{r}, a, ar$ be the three roots. Then, $\frac{a}{r} \cdot a \cdot ar = \frac{8}{27}$. Hence, $a^3 = \frac{8}{27} \Rightarrow a = \frac{2}{3}$ and $\frac{a}{r} + a + ar = -\frac{42}{27} \Rightarrow 3r^2 + 10r + 3 = 0 \Rightarrow r = -3, -\frac{1}{3}$ Thus the roots are $-\frac{2}{9}, \frac{2}{3}, -2$.

**Example 3.25** Solve $6x^3 - 11x^2 - 3x + 2 = 0$ whose roots are in H.P.
Solution : Let $\alpha, \beta, \gamma$ be the three roots in H.P. . Hence, $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in A.P. Thus, $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma} \Rightarrow 2\alpha\gamma = \alpha\beta + \beta\gamma$. Further we have $(\alpha\beta + \beta\gamma) + \gamma\alpha = -\frac{3}{6} \Rightarrow 2\alpha\gamma + \gamma\alpha = -\frac{1}{2} \Rightarrow 3\alpha\gamma = -\frac{1}{2} \Rightarrow \alpha\gamma = -\frac{1}{6}$. Moreover, $(\alpha\gamma)\beta = -\frac{2}{6} \Rightarrow -\frac{1}{6}\beta = -\frac{1}{3} \Rightarrow \beta = 2$. Now $\alpha + \beta + \gamma = \frac{11}{6} \Rightarrow \alpha + \gamma = \frac{11}{6} - 2 = -\frac{1}{6}$.
The equations $\alpha\gamma = -\frac{1}{6}$ and $\alpha + \gamma = -\frac{1}{6}$ together give us
$\alpha = -\frac{1}{2}$ and $\gamma = \frac{1}{3}$. Thus the three roots $\alpha, \beta, \gamma$ are $-\frac{1}{2}, 2, \frac{1}{3}$.

**Example 3.26** Find the sum of the squares of the roots of

$$2x^4 - 8x^3 + 6x^2 - 3 = 0.$$

Solution : Let $a, b, c, d$ be the four roots. Then,
$a + b + c + d = 4$, $ab + ac + ad + bc + bd + cd = 3$ and
$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 - 2(ab + ac + ad + bc + bd + cd)$
$\Rightarrow a^2 + b^2 + c^2 + d^2 = 4^2 - 2(3) = 10$.

**Theorem 3.6 (Fundamental Theorem of Algebra)** Every non-constant polynomial in $\mathbb{C}[x]$ has at least one root in $\mathbb{C}$.

**Remark** : If $f(x) \in \mathbb{C}[x]$ is a non-constant then by Fundamental Theorem of Algebra, $\exists\ \alpha_1 \in \mathbb{C}$ such that $f(\alpha_1) = 0$. Thus by Factor theorem $(x-\alpha_1)$ is a factor of $f(x)$ and we can write $f(x) = (x-\alpha_1)g_1(x)$, where $g_1(x) \in \mathbb{C}[x]$. Now applying Fundamental Theorem of Algebra to $g_1(x), \exists\ \alpha_2 \in \mathbb{C}$ such that $g_1(\alpha_2) = 0$ and hence $g_1(x) = (x-\alpha_2)g_2(x)$, so that $f(x) = (x-\alpha_1)g_1(x) = (x-\alpha_1)(x-\alpha_2)g_2(x)$. Continuing in this way, we get $f(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$, where $a$ is the leading coefficient and all $\alpha_i$'s need **not** be distinct.

**Theorem 3.7** If $f(x) \in \mathbb{R}[x]$ is a non-constant polynomial with a root $a+ib$ then $a-ib$ is also a root of $f(x)$.
Proof : Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, $a_i \in \mathbb{R}$, $a_n \neq 0$, $n \geq 1$. Let $\alpha = a+ib$ be the given root of $f(x)$, so that $f(\alpha) = 0 \Rightarrow a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$. Taking conjugate on both the sides we get $\overline{a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0} = \overline{0} = 0 \Rightarrow \overline{a_n\alpha^n} + \overline{a_{n-1}\alpha^{n-1}} + \cdots + \overline{a_1\alpha} + \overline{a_0} = 0$. But $a_i \in \mathbb{R}$ implies $\overline{a_i} = a_i$, for $i$. Thus we get, $a_n\overline{\alpha}^n + a_{n-1}\overline{\alpha}^{n-1} + \cdots + a_1\overline{\alpha} + a_0 = 0 \Rightarrow f(\overline{\alpha}) = 0 \Rightarrow f(a-ib) = 0$. Hence $a-ib$ is root of $f(x) = 0$, as desired.

**Corollary** : Any odd degree polynomial with real coefficients has at least one real root.
Proof : Let $f(x) \in \mathbb{R}[x]$ be of odd degree $n$ (say). By Fundamental theorem of Algebra, $f(x)$ has exactly $n$ roots. Pairing the complex roots with their conjugates, we see that atleast one root $\alpha$ (say) must be conjugate of itself (since degree of $f(x)$ is odd) i.e. $\alpha = \overline{\alpha} \Rightarrow \alpha$ is real root of $f(x)$, as desired.

## 3.6 Exercise

1. Let $p(x) = x^7 - 3x + 2$ and $q(x) = x^7 + 4x + 3$. Find $p(x) + q(x)$, $p(x)q(x)$.

2. Let $p(x) = 1 + x - x^2$ and $q(x) = x + \frac{1}{3}$. Find $p(x)q(x)$.



3. Find the roots of the following equations :

   (a) $x^4 - x^3 - 9x^2 - 11x - 4 = 0$.

   (b) $x^3 - 7x - 6 = 0$.

   (c) $x^3 - 6x^2 + 11x - 6 = 0$.

   (d) $4x^4 - 7x^3 - 5x^2 + 7x + 4 = 0$.

   (e) $(2x^2 - 3x + 1)(2x^2 + 5x + 1) = 9x^2$.

   (f) $(x+2)(x+3)(x+8)(x+12) = 4x^2$.

4. If $x^2 - hx - 21 = 0$ and $x^2 - 3hx + 35 = 0$ ($h > 0$) have a common root then show that $h = 4$.

5. Let $n \in \mathbb{N}$ and $\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + a_n = 0$, where $a_i \in \mathbb{R}$, $0 \leq i \leq n$. Define $f(x) = \frac{a_0}{n+1}x^{n+1} + \frac{a_1}{n}x^n + \frac{a_{n-1}}{2}x^2 + a_nx$. Show that $f(0) = f(1)$.

6. If $p(x), q(x) \in \mathbb{R}[x]$ such that $p(x)|q(x)$ and $q(x)|r(x)$. Show that $p(x)|r(x)$.

7. Let $p(x) = x - 2$ and $q(x) = 2x - 4 \in \mathbb{R}[x]$. Show that $p(x)|q(x)$ and $q(x)|p(x)$.

8. Let $p(x)$ and $q(x)$ be nonzero polynomials in $\mathbb{R}[x]$. Suppose there exists $c \neq 0$ such that $p(x) = cq(x)$ then show that $p(x)|q(x)$ and $q(x)|p(x)$. (Converse of Theorem 1)

9. Let $f(x) = x^3 - 2x^2 + 3x - 7$ and $g(x) = x^2 + 2$. Find $q(x), r(x) \in \mathbb{R}[x]$ such that $f(x) = g(x)q(x) + r(x)$.

10. Let $f(x) = x^3 - 8$ and $g(x) = x^2 + 2x + 4$. Find $q(x), r(x) \in \mathbb{R}[x]$ such that $f(x) = g(x)q(x) + r(x)$.

11. Use Factor Theorem to determine whether $g(x) = x + 1$ is a factor of $f(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$ or not.

12. Use Factor Theorem to determine whether $g(x) = x - 2$ is a factor of $f(x) = x^3 + 2x^2 - 3$ or not.

13. Use Factor Theorem to show that $x+1$ is a factor of $x^4-2x^3-3$.

14. Use Remainder Theorem to compute the remainder when $f(x)=x^5+x^4+x^3+x^2+x+1$ is divided by $g(x)=x+1$.

15. Use Remainder Theorem to compute the remainder when $f(x)=x^2+x+1$ is divided by $g(x)=x-1$.

16. Find the value of $a$, if $x-2$ is a factor of $x^4-x^3+ax^2+8=0$.

17. Find the greatest common divisor of the following polynomials in $\mathbb{R}[x]$:
    (a) $f(x)=x^3-3x^2+2x-6$, $g(x)=x^3-2x^2-2x-3$.
    (b) $f(x)=x^4-x^3+3x^2-3x$, $g(x)=x^3-x$.
    (c) $f(x)=x^3-1$, $g(x)=x^2-2x+3$.

18. Check whether $f(x)=x^5-x^4+x^3-x^2+x-1$ and $g(x)=x^2+x+1$ are relatively prime in $\mathbb{R}[x]$.

19. Show that $(f(x),g(x))=(x^4+x^3+2x^2+x+1, x^3-1)=x^2+x+1$. Further find $m(x),n(x)\in\mathbb{R}[x]$ such that $x^2+x+1=m(x)f(x)+n(x)g(x)$.

20. Solve $x^3-3x^2+4=0$, two of its roots being equal.

21. Solve $x^3-5x^2-4x+20=0$given that the difference of its two roots is 3.

22. Solve $2x^3-x^2-22x-24=0$, two of the roots being in the ratio $3:4$.

23. Solve $x^3-12x^2+39x-28=0$, the roots being in A.P.

24. Solve $x^3-9x^2+23x-16=0$, the roots being in A.P.

25. Solve $3x^3-26x^2+52x-24=0$, the roots being in G.P.

26. Solve $105x^3-142x^2+60x-8=0$, the roots being in H.P.



27. Solve $x^3-5x^2-2x+24=0$ given that product of two roots is 12.

28. Solve $x^3-5x^2-16x+80=0$, if the sum of the two roots is zero.

29. Find the sum of squares of roots of $2x^4-6x^3+5x^2-7x+1=0$.

**Solution and Hints**

1. $p(x)+q(x)=2x^7+x+5$, and $p(x)q(x)=x^{14}+x^8+5x^7-12x^2-x+6$.

2. $p(x)q(x)=\frac{1}{3}+\frac{4}{3}x+\frac{2}{3}x^2-x^3$.

3. (a) $4,-1-1-1$ (b) $-1,-2,3$ (c) $1,2,3$
   (d) Divide the equation by $x^2$ and use the substitution $x-\frac{1}{x}=y$ and simplify to get the roots $x=\frac{1\pm\sqrt{5}}{2},\frac{3\pm\sqrt{73}}{8}$.
   (e) $x=0$ is not a root, dividing by $x$ and substituting $y=2x+\frac{1}{x}$ and simplifying we get $x=\frac{-3\pm\sqrt{7}}{2},\frac{2\pm\sqrt{2}}{2}$.
   (f) $(x^2+14x+24)(x^2+11x+24)=4x^2$ and now divide by $x^2$ and use a suitable substitution to get $x=\dfrac{-15\pm\sqrt{129}}{2}$, $-6,-4$

4. Let $c$ be a common root of both the equations, subtracting we get $c=\frac{28}{h}$, substituting value of $c$ in first equation we get $h=\pm 4$.

9. $q(x)=x-2$, $r(x)=x-3$. 10. $q(x)=x-2, r(x)=0$

11. Yes 12. No 14. $0$. 15. $3$. 16. $a=-4$

17. (a) $d(x)=x-3$ (b)$d(x)=x^2-x$ (c) $d(x)=1$

19. $m(x)=\frac{1}{2}$, $n(x)=-\frac{x+1}{2}$. 20. $2,2,-1$ 22. $-\frac{3}{2},-2,4$

23. $1,4,7$ 24. $1,3,5$ 25. $6,2,\frac{2}{3}$ 27. $-2,3,4$ 29. $4$

# Chapter 4

# Matrices

## 4.1 Introduction

**Definition 4.1 (Matrix)** A matrix is a rectangular array of numbers with $m$ rows and $n$ columns. It is represented by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n},$$

where $a_{ij} \in \mathbb{R}$ ($1 \leq i \leq m$, $1 \leq j \leq n$) are called **entries**. We say that matrix $A$ is of size $m \times n$ in this case.

For simplicity we shall also denote the above matrix by $A = (a_{ij})$.

**Definition 4.2 (Row or Column matrix)** A matrix of order $1\times n$ is called a row matrix and a matrix of order $m \times 1$ is called column matrix.

For instance, $A = \begin{bmatrix} -2 & 3 & 0 \end{bmatrix}$ is a row matrix of size $1\times 3$ and $B = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 13 \end{bmatrix}$ is a column matrix of size $4 \times 1$.

**Definition 4.3 (Square matrix)** A matrix having same number of rows and columns is called a sqaure matrix.

**Definition 4.4 (Zero matrix or Null matrix)**
A matrix containing all entries equal to zero is called zero matrix.





**Definition 4.5 (Diagonal entries of a matrix)** Let $A = (a_{ij})$ be an $n \times n$ square matrix. The entries $a_{ii}, 1 \leq i \leq n$ are called diagonal entries of the matrix $A$.

**Definition 4.6 (Diagonal matrix)** A square matrix of size $n \times n$ ($n \in \mathbb{N}$) whose all non-diagonal elements are 0 is called a diagonal matrix.

**Definition 4.7 (Identity matrix)** A square matrix of size $n \times n$ ($n \in \mathbb{N}$) whose all diagonal entries are 1 and non-diagonal elements are 0 is called identity matrix of size $n$. We will denote it by $I_n$.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**Definition 4.8 (Addition and scalar multiplication)**
Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of size $m \times n$. Then the sum of two matrices $A$ and $B$ is defined as $A + B = (a_{ij} + b_{ij})$. Further for any $k \in \mathbb{R}$, we define $kA = (ka_{ij})$, the scalar multiplication.

**Definition 4.9 (Matrix multiplication)** Let $A = (a_{ij})$ be a matrix of size $m \times p$ and $B = (b_{ij})$ be a matrix of size $p \times n$. Then the product of two matrices $A$ and $B$ is a matrix of size $m \times n$ which is defined as $AB = (c_{ij}) = \sum_{k=1}^{p} a_{ik}b_{kj}$.

**Definition 4.10 (Elementary row - transformations)**
We define elementary row - transformations as follows :

1. Replacing $i$th row of $A$ by a scalar multiple (say $c \neq 0$) of the same row i.e. in notation, $cR_i$.
2. Replacing $i$th row by $i$th row plus $\lambda$ times the $j$th row i.e. in notation $R_i \to R_i + \lambda R_j$.
3. Interchanging $i$th row with $j$th row i.e. in notation $R_i \leftrightarrow R_j$.

**Definition 4.11 (Elementary column - transformations)**
We define elementary column - transformations as follows :

1. Replacing $i$th column of $A$ by a scalar multiple (say $c \neq 0$) of the same column i.e. in notation, $cC_i$.

2. Replacing $i$th column by $i$th column plus $\lambda$ times the $j$th column i.e. in notation $C_i \to C_i + \lambda C_j$.

3. Interchanging $i$th column with $j$th column i.e. in notation $C_i \leftrightarrow C_j$.

**Example 4.1** We now apply elementary row - transformations to reduce matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ to the identity matrix. So we perform the following elementary row operations :

$R_2 \to -2R_1 + R_2, R_3 \to -R_1 + R_3$ to get, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{bmatrix}$

$R_3 \to R_3 + 2R_2$ to get, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$

$R_3 \to -R_3$ to get, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to 3R_3 + R_2, R_1 \to -3R_3 + R_1$ to get, $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \to -2R_2 + R_1$ to get, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the Identity matrix $I_3$.

**Definition 4.12 (Row echelon form)** A matrix is said to be in row echelon form if it satisfies the following properties :

1. If a row has a nonzero entry and if this nonzero entry is 1, then we call this a **leading 1**.



2. Rows that consist of all zeros must be grouped together at the bottom.

3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

**Example 4.2** The following matrices are in row echelon form:
$\begin{bmatrix} 1 & 5 & 2 & -7 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

**Example 4.3** Use elementary row operations to convert the matrix in its row echelon form : $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix}$.

Solution : Perform $R_2 \leftrightarrow R_3$ to get, $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Perform $\frac{1}{2}R_2$ and $\frac{1}{3}R_3$ to get, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Perform $R_1 \leftrightarrow R_2$ to get, $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the required row echelon form.

**Example 4.4** Use elementary row operations to convert the matrix in its row echelon form $\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 2 & 3 & 1 \end{bmatrix}$.

Solution : Perform $R_2 \leftrightarrow R_3$, $\frac{1}{3}R_1$ to get, $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ Perform $R_1 \leftrightarrow R_2$ to get, $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Perform $\frac{1}{2}R_1$, $\frac{1}{4}R_3$ to get, $\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the required row echelon form.

**Definition 4.13 (Rank of a matrix)** The number of nonzero rows in the row echelon form of matrix $A$ is called the rank of matrix $A$. It is denoted by $\rho(A)$.

**Example 4.5** Find the rank of $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$.

Solution : We now reduce the matrix to its echelon form.

Perform $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$ to get,

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}.$$

Perform $R_2 \to R_2 - R_3$ to get, $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Perform $-\frac{1}{2}R_2$ to get, $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus we have reduced A in its echelon form which has two nonzero rows. Hence $\rho(A) = 2$.

**Definition 4.14 (Reduced row echelon form)** A matrix is said to be in reduced row echelon form if :

1. It is in row echelon form.

2. Each column that contains a leading 1 has zeros everywhere else in that column.

**Example 4.6** The matrices given below are in reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$



**Example 4.7** Find row echelon form and reduced row echelon form of $\begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 1 & 0 & 1 \end{bmatrix}$.

Solution : Perform $R_2 \leftrightarrow R_3$ and $\frac{1}{2}R_1$ to get, $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Perform $\frac{1}{3}R_3$, $R_1 \leftrightarrow R_2$ to get, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, which is row echelon form.

Further perform $R_2 \to R_2 - 2R_3$ and $R_1 \to R_1 - R_3$ to get,

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is now reduced row echelon form.

## 4.2 System of Linear equations

**Definition 4.15 (System of Linear equations)** A finite set of linear equations in the variables $x_1, x_2, \cdots, x_n$ is called a **system of linear equations** or a **linear system**. The sequence of number $s_1, s_2, \cdots, s_n$ is called a **solution** of the system if $x_1 = s_1$, $x_2 = s_2$, $\cdots$, $x_n = s_n$ is a solution of every equation in the system.

A system of equations is said to be **consistent** if there is at least one solution of the system, and called **inconsistent** if it has no solution.

We shall later discuss in details the conditions of consistency for a linear system.

Consider an arbitrary system of $m$ linear equations in $n$ unknowns given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

where $a_{ij}, b_k \in \mathbb{R}, 1 \le i, k \le m, 1 \le j \le n$

Now we can write this system in the form of $AX = B$ i.e.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}.$$

**Definition 4.16 (Augmented matrix)** Consider the system $AX = B$ as stated above. The matrix of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called augmented matrix for the system and is denoted by $[A|B]$.

**Example 4.8** The system of linear equations given by

$$\begin{aligned} 2x + y - z + 3w &= 8, \\ x + y + z - w &= -2, \\ 3x + 2y - z &= 6, \\ 4y + 3z + 2w &= -8 \end{aligned}$$



can be written in the form as $AX = B$

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & -1 \\ 3 & 2 & -1 & 0 \\ 0 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 6 \\ -8 \end{bmatrix}$$

**Example 4.9** Write the following linear system in $AX = B$ form.

$$\begin{aligned} x + 2y - 3z + 4w &= 2 \\ 2x + 5y - 2z + w &= 1 \\ 5x + 12y - 7z + 6w &= 7. \end{aligned}$$

Solution : Write the system as

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & 5 & -2 & 1 \\ 5 & 12 & -7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

**Example 4.10** Find the solution set of the linear equation $7x - 5y = 3$. Solution : Let $y = t \in \mathbb{R}$ be arbitrary. Then $x = \dfrac{3+5t}{7}$. Thus the solution set is given by

$$\left\{ (x, y) \in \mathbb{R}^2 : x = \frac{3+5t}{7}, y = t, t \in \mathbb{R} \right\}.$$

**Example 4.11** Find the solution set of the linear equation

$$x + y + z - w + 2u = 0.$$

**Solution:** Let $u = t \in \mathbb{R}$, $w = s \in \mathbb{R}$, $z = r \in \mathbb{R}$, $y = m \in \mathbb{R}$ to get $x = -m - r + s - 2t$. Thus the solution set is given by

$$\{(-m - r + s - 2t, m, r, s, t) : m, r, s, t \in \mathbb{R}\}.$$

**Example 4.12** Consider the system given by

$$\begin{aligned} x - 2y + z &= 3, \\ 2x + 3y + z &= 4, \\ y - 7z &= 5, \\ z - y - z &= 2. \end{aligned}$$

This system can be written as $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & 1 \\ 0 & 1 & -7 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 2 \end{bmatrix}$
and hence the augmented matrix

$$[A|B] = \begin{bmatrix} 1 & -2 & 3 & 3 \\ 2 & 3 & 1 & 4 \\ 0 & 1 & -7 & 5 \\ 1 & -1 & -1 & 2 \end{bmatrix}.$$

**Definition 4.17** A system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n. \end{aligned}$$

where $a_{ij} \in \mathbb{R}, 1 \le i \le m, 1 \le j \le n$ is said to be **homogeneous** if all the $b_i$'s are all zero. Further, if some $b_i$ is nonzero, then the system is said to be a **non-homogeneous system of linear equations.**

**Example 4.13**

$$\begin{aligned} 2x + y + 3z + 6w &= 0, \\ 3x - y + z + 3w &= 0, \\ -x - 2y + 3z &= 0, \\ -x - 4y - 2z - 7w &= 0 \end{aligned}$$



is a system of homogeneous linear equations, whereas

$$\begin{aligned} 2x - y - z &= 0, \\ x + 2y + z &= 0, \\ 4x - 7y - 5z &= -1. \end{aligned}$$

is a system of non homogeneous linear equations.

**Definition 4.18 (Leading variables and free variables)**
Let $AX = B$ be a system of linear equations. The variables corresponding to leading 1's in the row echelon form of the augmented matrix $[A|B]$ are called **leading variables** or **pivots**. The nonleading variables are called **free variables**.

**Example 4.14** Identify the free variables and leading variables for the following system:

$$\begin{aligned} 2x + 6y &= -11, \\ 6x + 20y - 6z &= -3, \\ 6y - 18z &= -1. \end{aligned}$$

Solution : We write the system in the form of
$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$
The augmented matrix is

$$[A|B] = \begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}$$

Perform $\frac{1}{6}R_3$, $\frac{1}{2}R_1$, $R_2 \to R_2 - 3R_1$,

$$\begin{bmatrix} 1 & 3 & 0 & -\frac{11}{2} \\ 0 & 2 & -6 & 30 \\ 0 & 1 & -3 & -\frac{1}{6} \end{bmatrix}$$

Perform $R_2 \to R_2 - 2R_3$, $\begin{bmatrix} 1 & 3 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 & \frac{91}{3} \\ 0 & 1 & -3 & -\frac{1}{6} \end{bmatrix}$.

Perform $R_2 \leftrightarrow R_3$, $\begin{bmatrix} 1 & 3 & 0 & -\frac{11}{2} \\ 0 & 1 & -3 & -\frac{1}{6} \\ 0 & 0 & 0 & \frac{91}{3} \end{bmatrix}$ which is the row-echelon form of the augmented matrix $[A|B]$. The variables corresponding to leading 1's are $x$ and $y$. The variable $z$ is free.

**Example 4.15** Identify the free variables and leading variables for the following system:

$$\begin{aligned} x + 2y - 3z - 4w &= 2, \\ 2x + 4y - 5z - 7w &= 7, \\ -3x - 6y + 11z + 14w &= 0. \end{aligned}$$

Solution : We write the system in the form

$$\begin{bmatrix} 1 & 2 & -3 & -4 \\ 2 & 4 & -5 & -7 \\ -3 & -6 & 11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$$

Thus the augmented matrix is

$$[A|B] = \begin{bmatrix} 1 & 2 & -3 & -4 & 2 \\ 2 & 4 & -5 & -7 & 7 \\ -3 & -6 & 11 & 14 & 0 \end{bmatrix}$$

Perform $R_2 \to R_2 - 2R_1$ and $R_3 \to 3R_1 + R_3$ to get,

$\begin{bmatrix} 1 & 2 & -3 & -4 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 & 6 \end{bmatrix}$. Perform $R_3 \to R_3 - 2R_2$ to get,

$\begin{bmatrix} 1 & 2 & -3 & -4 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is the row-echelon form of the augmented matrix $[A|B]$. The variables corresponding to leading 1's are $x$ and $z$. The variables $y$ and $w$ are free.



## 4.3 Elimination Methods

**Gauss Elimination Method:** Suppose we have a system of linear equations $AX = B$. The following steps are used to find the solutions of this system by Gauss Elimination method :

1. Reduce the augmented matrix $[A|B]$ to its **row echelon form** using elementary row transformations.

2. Identify the free variables and assign them values in $\mathbb{R}$.

3. Form equations from the obtained row echelon form and find the expression for the remaining leading variables in terms of free variables.

**Gauss Jordan Method** : Consider a system of linear equations $AX = B$. The following steps are used to find the solutions of this system by Gauss Jordan method :

1. Reduce the augmented matrix $[A|B]$ to its **reduced row echelon form** using elementary row transformations.

2. Identify the free variables and assign them values in $\mathbb{R}$.

3. Form equations from the obtained row echelon form and find the expression for the remaining leading variables in terms of free variables.

**Example 4.16** Solve the following system by using Gauss Elimination method:

$$\begin{aligned} x + y + 2z &= 9, \\ 2x + 4y - 3z &= 1, \\ 3x + 6y - 5z &= 0. \end{aligned}$$

Solution : Reduce the augmented matrix to row echelon form

$$[A|B] = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Perform $R_2 \to R_2 - 2R_2$ and $R_3 \to R_3 - 3R_1$ to get,

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Perform $\frac{1}{2}R_2$ to get, $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$

Perform $R_3 \to R_3 - 3R_2$ to get, $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$

Perform $-2R_2$ to get, $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$

so that $[A|B]$ is in row echelon form. Thus, we get the following equations : $x + y + 2z = 9$, $x - \frac{7}{2}y = -\frac{17}{2}$, $z = 3$.

Hence substituting the value $z = 3$ we get $y = 2$, which further yields $x = 1$. Hence $x = 1$, $y = 2$, $z = 3$ is the solution of the given system.

**Example 4.17** Solve the following system by using Gauss Jordan method:

$$\begin{aligned} x + 3y - 2z + 2u &= 0, \\ 2x + 6y - 5z - 2w + 4u - 3v &= -1, \\ 5z + 10w + 15v &= 0, \\ 2x + 6y + 8w + 4u + 18v &= 0. \end{aligned}$$

**Solution:** Reduce the augmented matrix to **reduced** row echelon form

$$[A|B] = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Perform $R_2 \to R_2 - 2R_1$ and $R_4 \to R_4 - 2R_1$ to get,

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$



Perform $R_3 \to R_3 + 5R_2$ and $R_4 \to R_4 + 4R_2$ to get,

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Perform $\frac{1}{6}R_4 \leftrightarrow R_3$ to get row echelon form,

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Perform $R_2 \to R_2 - 3R_1$ and $R_1 \to R_1 + 2R_2$ to get **reduced** row echelon form,

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the leading variables are $x, z, v$ and the free variables are $y, u, w$. So put $y = t$, $u = s$, $w = r$ where $t, s, r \in \mathbb{R}$ are arbitrary. Forming equations from reduced row echelon form, we get,

$$x + 3y + 4w + 2u = 0, z + 2w = 0, v = \frac{1}{3}.$$

Using values of $y, u, w$ and $v$ we get,

$x = -3t - 4r - 2s$, $y = t, z = -2r$, $w = r$, $u = s$, $v = \frac{1}{3}$ as the solution of the system.

## 4.4 Consistency of a system

Consider a system of $m$ linear equations in $n$ unknowns given by,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Let the system be written in the form of $AX = B$ (say).

1. **Consistent system** : The system $AX = B$ is said to be consistent if $\rho(A) = \rho(A, B)$, i.e. the system has atleast one solution.

2. **Unique solution** : The system $AX = B$ admits a unique solution if $\rho(A) = \rho(A|B) = n$ (number of unknowns).

3. **No solution** : If $\rho(A) \neq \rho(A|B)$ then the system is inconsistent and admits no solution.

4. **Infinite number of solutions** : If $\rho(A) = \rho(A|B) < n$ (number of unknowns), then the system admits an infinite number of solutions.

**Particular cases :**

1. **Homogeneous system of linear equations** $AX = O$:

   (a) For system AX = **O**, the identity $\rho(A) = \rho(A|B)$ holds clearly. Thus we conclude that a system of homogeneous equations is always consistent.

   (b) $X = (x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ is always a solution of a system of homogeneous equations, which will be called the **trivial solution** henceforward.

   (c) If $\rho(A) < n$ (number of unknowns) then the system admits an infinite number of solutions.

2. **System** $AX = B$, **where** $A$ **is an** $n \times n$ **matrix** :

   (a) The system $AX =$ **O** admits only trivial solution number of solutions if $\det(A) \neq 0$ or equivalently $\rho(A) = n$, since in this case $X = A^{-1}$**O** $=$ **O**.

   (b) The system $AX =$ **O** admits infinite number of solutions if $\det(A) = 0$.

   (c) $AX = B$, a system of nonhomogeneous linear equations either admits infinite number of solutions or no solution if $det(A) = 0$ but if $det(A) \neq$ **O**, then the system admits a unique solution viz $X = A^{-1}B$.



**Example 4.18** Solve the system

$$\begin{aligned} 2x - y - 3z &= 1, \\ 5x + 2y - 6z &= 5, \\ 3x - y - 4z &= 7. \end{aligned}$$

Solution : Write the system as

$$\begin{bmatrix} 2 & 1 & -3 \\ 5 & 2 & -6 \\ 3 & -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$$

Thus, $[A|B] = \begin{bmatrix} 2 & 1 & -3 & 1 \\ 5 & 2 & -6 & 5 \\ 3 & -1 & -4 & 7 \end{bmatrix}$.

Perform $R_2 \to -5R_1 + 2R_2$, $R_3 \to -3R_1 + 2R_3$ to get,

$$\begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & -1 & 3 & 5 \\ 0 & -5 & 1 & 11 \end{bmatrix}$$

Perform $R_3 \to -5R_2 + R_3$, $\begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & -14 & -14 \end{bmatrix}$

Here $\rho(A) = \rho(A|B) = 3$ (number of variables), so the system has a unique solution, $z = \frac{-14}{-14} = 1$.

Now $-y + 3z = 6$ gives $y = -2$ and $2x + y - 3z = 1$ gives $x = 3$. The unique solution of the system is given by $x = 3, y = -2, z = 1$.

**Example 4.19** Solve the system

$$\begin{aligned} 2x + y - z + 3w &= 8, \\ x + y + z - w &= -2, \\ 3x + 2y - z &= 6, \\ 4y + 3z + 2w &= -8 \end{aligned}$$

Solution : Write the system as $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & -1 \\ 3 & 2 & -1 & 0 \\ 0 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 6 \\ -8 \end{bmatrix}$.

Thus, $[A|B] = \begin{bmatrix} 2 & 1 & -1 & 3 & 8 \\ 1 & 1 & 1 & -1 & -2 \\ 3 & 2 & -1 & 0 & 6 \\ 0 & 4 & 3 & 2 & -8 \end{bmatrix}$

Perform $R_1 \leftrightarrow R_2$ to get, $\begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 2 & 1 & -1 & 3 & 8 \\ 3 & 2 & -1 & 0 & 6 \\ 0 & 4 & 3 & 2 & -8 \end{bmatrix}$

Perform $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$ to get,

$$\begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -3 & 5 & 12 \\ 0 & -1 & -4 & 3 & 12 \\ 0 & 4 & 3 & 2 & -8 \end{bmatrix}$$

Perform $R_3 \to R_3 - R_2, R_4 \to R_4 + 4R_2$ to get,

$$\begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -3 & 5 & 12 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -9 & 22 & 40 \end{bmatrix}$$

Perform $R_4 \to R_4 - 9R_3$ to get,

$$\begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -3 & 5 & 12 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 40 & 40 \end{bmatrix}$$

Here $\rho(A) = \rho(A|B) = 4$ (number of variables), so the system has a unique solution. Here, $40w = 40 \Rightarrow w = \frac{40}{40} = 1$.

Now $-z - 2w = 0$ gives $z = -2$. Further $-y - 3z + 5w = 12$ gives $y = -1$ and $x + y + z - w = -2$ gives $x = 2$.



The unique solution of the system is given by $x = 1$, $y = -1$, $z = -2$, $w = 1$.

**Example 4.20** Examine consistency of the system :

$$\begin{aligned} 2x + 6y &= -11, \\ 6x + 20y - 6z &= -3, \\ 6y - 18z &= -1. \end{aligned}$$

Solution : Write the system in the form of

$$[A|B] = \begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}.$$

Perform $\frac{1}{6}R_3$, $\frac{1}{2}R_1$, $R_2 \to R_2 - 3R_1$, $\begin{bmatrix} 1 & 3 & 0 & -\frac{11}{2} \\ 0 & 2 & -6 & 30 \\ 0 & 1 & -3 & -\frac{1}{6} \end{bmatrix}$.

Perform $R_2 \to R_2 - 2R_3$, $\begin{bmatrix} 1 & 3 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 & \frac{91}{3} \\ 0 & 1 & -3 & -\frac{1}{6} \end{bmatrix}$.

Perform $R_2 \leftrightarrow R_3$, $\begin{bmatrix} 1 & 3 & 0 & -\frac{11}{2} \\ 0 & 1 & -3 & -\frac{1}{6} \\ 0 & 0 & 0 & \frac{91}{3} \end{bmatrix}$

Thus $\rho(A) = 2$ and $\rho(A|B) = 3$, hence $\rho(A) \neq \rho(A|B)$, so that the system is inconsistent and therefore has no solution.

**Example 4.21** Solve the system

$$\begin{aligned} x + 2y - 3z + 4w &= 2 \\ 2x + 5y - 2z + w &= 1 \\ 5x + 12y - 7z + 6w &= 7. \end{aligned}$$

Solution : Write the system as

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & 5 & -2 & 1 \\ 5 & 12 & -7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

Thus $[A|B] = \begin{bmatrix} 1 & 2 & -3 & 4 & 2 \\ 2 & 5 & -2 & 1 & 1 \\ 5 & 12 & -7 & 6 & 7 \end{bmatrix}$ Perform $R_2 \to R_2 - 2R_1$ and $R_3 \to -5R_1 + R_3$ to get,
$\begin{bmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & 1 & 4 & -7 & -3 \\ 0 & 2 & 8 & -14 & 3 \end{bmatrix}$. Perform $R_3 \to -2R_2 + R_3$ to get,
$\begin{bmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & 1 & 4 & -7 & -3 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$. Since $\rho(A) \neq \rho(A|B)$, the system is inconsistent and hence has no solution.

**Example 4.22** Solve the system of equations:

$$\begin{aligned} 2x - 3y + 5z &= 1, \\ 3x + y - z &= 2, \\ x + 4y - 6z &= 1. \end{aligned}$$

Solution : $[A|B] = \begin{bmatrix} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{bmatrix}$

Perform $R_1 \leftrightarrow R_3$ to get, $\begin{bmatrix} 1 & 4 & -6 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & -3 & 5 & 1 \end{bmatrix}$

Perform $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$ to get,

$$\begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{bmatrix}$$

Perform $R_3 \to R_3 - R_2$ to get, $\begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Since $\rho(A) = \rho(A|B) = 2 < n = 3$ (3 unknowns), the system has infinite solutions. Here $x, y$ are leading variables and $z$ is a free variable. Put



$z = t, t \in \mathbb{R}$ in $x + 4y - 6z = 1$ and $-11y + 17z = -1$ to get $y = \frac{17t+1}{11}$ and $x = \frac{7-2t}{11}$. Thus the infinite solutions of the system are given by

$$x = \frac{7-2t}{11}, y = \frac{17t+1}{11}, z = t \text{ where } t \in \mathbb{R}.$$

**Example 4.23** Solve the system of equations :

$$\begin{aligned} x + 2y - 3z - 4w &= 2, \\ 2x + 4y - 5z - 7w &= 7, \\ -3x - 6y + 11z + 14w &= 0. \end{aligned}$$

Solution : Write the system as

$$\begin{bmatrix} 1 & 2 & -3 & -4 \\ 2 & 4 & -5 & -7 \\ -3 & -6 & 11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$$

Thus $[A|B] = \begin{bmatrix} 1 & 2 & -3 & -4 & 2 \\ 2 & 4 & -5 & -7 & 7 \\ -3 & -6 & 11 & 14 & 0 \end{bmatrix}$

Perform $R_2 \to R_2 - 2R_1$ and $R_3 \to 3R_1 + R_3$ to get,

$$\begin{bmatrix} 1 & 2 & -3 & -4 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 & 6 \end{bmatrix}$$

Perform $R_3 \to R_3 - 2R_2$ to get, $\begin{bmatrix} 1 & 2 & -3 & -4 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Since $\rho(A) = \rho(A|B) = 2 < n = 4$ (4 unknowns), the system has infinite solutions. Here variables $x$, $z$ are leading and $y$, $w$ are free variables. So put $y = t$, ($t \in \mathbb{R}$) and $w = s$ ($s \in \mathbb{R}$) to get $x = 11 - 2t + s$ and $z = 3 + s$. Thus the system has infinitely many solutions given by

$x = 11 - 2t + s, y = t, z = 3 + s, w = s$ where, $t, s \in \mathbb{R}$.

**Example 4.24** Solve the system of equations:

$$\begin{aligned} x+2y-3z-2w+4u &= 1, \\ 2x+5y-8z-w+6u &= 4, \\ x+4y-7z+5w+2u &= 8. \end{aligned}$$

**Solution:** Write the system as

$$\begin{bmatrix} 1 & 2 & -3 & -2 & 4 \\ 2 & 5 & -8 & -1 & 6 \\ 1 & 4 & -7 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

Thus $[A|B] = \begin{bmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 4 & -7 & 5 & 2 & 8 \end{bmatrix}$

Perform $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_2$ to get,

$$\begin{bmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 2 & -4 & 7 & -2 & 7 \end{bmatrix}.$$

Perform $R_3 \to R_3 - 2R_2$, $R_1 \to R_1 + 2R_3$, $R_2 \to R_2 - 3R_3$, $R_1 \to R_1 - 2R_2$ sequentially to get, $\begin{bmatrix} 1 & 0 & 1 & 0 & 24 & 21 \\ 0 & 1 & -2 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$.

Since $\rho(A) = \rho(A|B) = 3 < n = 5$ (5 unknowns), the system has infinite solutions. Here variables $x$, $y$, $w$ are leading and $z$, $u$ are free variables. So put $z = t, (t \in \mathbb{R})$ and $u = s$ $(s \in \mathbb{R})$ to get $x = 21 - t + 24s$, $y = -7 + 2t + 8s$ and $w = 3 - 2s$. Thus the system has infinitely many solutions given by,

$x = 21 - t + 24s, y = -7 + 2t + 8s, z = t, w = 3 - 2s, u = s,$ where $t, s \in \mathbb{R}$.



**Example 4.25** Find the values of $\lambda$ for which the system will have a unique solution:

$$\begin{aligned} x+y+z &= 6, \\ x+2y+3z &= 10, \\ x+2y+\lambda z &= 10. \end{aligned}$$

Solution : Here $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & 10 \end{bmatrix}$

Perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to get,

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & 4 \end{bmatrix}$$

Perform $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$ to get,

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & 0 \end{bmatrix}$$

The system has a unique solution if $\rho(A) = \rho(A|B) = 3$ (3 unknowns). If $\lambda - 3 = 0$ then $\rho(A) = \rho(A|B) = 2 < 3$ (number of unknowns), which will lead to infinite solutions to the system. Hence the system has a unique solution if $\lambda \neq 3$.

**Example 4.26** For what values of $\lambda$ will the system admit no solution :

$$\begin{aligned} 2x+y &= \lambda, \\ x-z &= 1, \\ y+2z &= 1. \end{aligned}$$

Solution: Here $[A|B] = \begin{bmatrix} 2 & 1 & 0 & \lambda \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.

Perform $R_1 \leftrightarrow R_2$ to get,
$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & \lambda \\ 0 & 1 & 2 & 1 \end{bmatrix}$. Perform $R_2 \to R_2 - 2R_1$ to get, $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & \lambda-2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.

Perform $R_3 \to R_3 - R_2$ to get, $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & \lambda-2 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}$ .

Here $\rho(A) = 2$. If $\lambda - 3 = 0$ then $\rho(A|B) = 2$ and in this case the system admits infinite solutions. But if $\lambda - 3 \neq 0 \Rightarrow \rho(A) \neq \rho(A|B) = 3$. Hence $\lambda \neq 3$ leads to no solution.

**Example 4.27** For what values of $\lambda$ does the system admit infinite solutions :

$$\begin{aligned} 2x - 3y + 6z - 5w &= 3, \\ y - 4z + w &= 1, \\ 4x - 5y + 8z - 9w &= \lambda. \end{aligned}$$

Solution : Here $[A|B] = \begin{bmatrix} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 4 & -5 & 8 & -9 & \lambda \end{bmatrix}$ .

Perform $R_3 \to R_3 - 2R_1$ to get, $\begin{bmatrix} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & -4 & 1 & \lambda-6 \end{bmatrix}$ .

Perform $R_3 \to R_3 - R_2$ to get, $\begin{bmatrix} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & \lambda-7 \end{bmatrix}$ .

Here $\rho(A) = 2$. If $\lambda - 7 = 0$ then $\rho(A|B) = 2$ and in this case the system admits infinite solutions. Thus $\lambda = 7$ leads to infinite solutions.

**Example 4.28** Determine the values of $k$ so that the system

$$\begin{aligned} x - 2y &= 1, \\ x - y + kz &= -2, \\ ky + 4z &= 6. \end{aligned}$$



(i) has a unique solution, (ii) no solution, (iii) infinite number of solutions.

Solution : Here $[A|B] = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 1 & -1 & k & -2 \\ 0 & k & 4 & 6 \end{bmatrix}$

Perform $R_2 \to R_2 - R_1$ to get, $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & k & -3 \\ 0 & k & 4 & 6 \end{bmatrix}$

Perform $R_3 \to R_3 - kR_2$ to get, $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & k & -3 \\ 0 & 0 & 4-k^2 & 6+3k \end{bmatrix}$

(i) Unique solution : The system admits a unique solution if $4 - k^2 \neq 0$ i.e. $k \neq 2$ and $k \neq -2$.

(ii) No solution : The system admits no solution if $k = 2$, since $4 - k^2 = 0$ but $6 + 3k \neq 0$, so that $\rho(A) \neq \rho(A|B)$.

(iii) Infinite solutions : The system admits infinite solutions if $k = -2$ since $4 - k^2 = 0$ and $6 + 3k = 0$, so that $\rho(A) = \rho(A|B) = 2 < 3$ (number of unknowns).

**Example 4.29** Determine the values of $k$ so that the system

$$\begin{aligned} x + y - z &= 1, \\ 2x + 3y + kz &= 3, \\ x + ky + 3z &= 2. \end{aligned}$$

(ii)has a unique solution, (ii) no solution, (iii) infinite number of solutions.

Solution : Here $[A|B] = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{bmatrix}$ . Perform $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$ to get,

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{bmatrix}$$

Perform $R_3 \to R_3 - (k-1)R_2$ to get,

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & (3+k)(2-k) & 2-k \end{bmatrix}$$

(i) Unique solution : The system admits a unique solution if $(3+k)(2-k) \neq 0$ i.e. $k \neq 2$ and $k \neq -3$.

(ii) No solution : The system admits no solution if $k = -3$, since $(3+k)(2-k) = 0$ but $2-k = 5 \neq 0$, so that $\rho(A) \neq \rho(A|B)$.

(iii) Infinite solutions : The system admits infinite solutions if $k = 2$ since $(3+k)(2-k) = 0$ and $2-k = 0$, so that $\rho(A) = \rho(A|B) = 2 < 3$ (number of unknowns).

**Example 4.30** Determine the values of $k$ so that the system

$$\begin{aligned} x + y + kz &= 1, \\ x + ky + z &= 1 \\ kx + y + z &= 1. \end{aligned}$$

(i) has a unique solution, (ii) no solution, (iii) infinite number of solutions.

Solution : Here $[A|B] = \begin{bmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{bmatrix}$

Perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - kR_1$ to get,

$$\begin{bmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1+k & 1-k^2 & 1-k \end{bmatrix}$$

Perform $R_3 \to R_3 + R_2$ to get,

$$\begin{bmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & (2+k)(1-k) & 1-k \end{bmatrix}$$



(i) Unique solution : The system admits unique solution if $(2+k)(1-k) \neq 0$ i.e. $k \neq -2$ and $k \neq 1$.

(ii) No solution : The system admits no solution if $k = -2$, so that $\rho(A) \neq \rho(A|B)$.

(iii) Infinite solutions : The system admits infinite solutions if $k = 1$, so that $\rho(A) = \rho(A|B) = 2 < 3$ (number of unknowns).

**Example 4.31** Determine the values of $k$ so that the system

$$\begin{aligned} x + 2y + z &= 3, \\ x + y + z &= k, \\ 3x + y + 3z &= k^2. \end{aligned}$$

(i) has a unique solution, (ii) no solution, (iii) infinite number of solutions.

Solution : Here $[A|B] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & k \\ 3 & 1 & 3 & k^2 \end{bmatrix}$. Perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$ to get,

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & k-3 \\ 0 & -5 & 0 & k^2-9 \end{bmatrix}$$

Perform $R_3 \to R_3 - 5R_2$ to get, $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & k-3 \\ 0 & 0 & 0 & k^2-5k+6 \end{bmatrix}$

(i) Unique solution : The system cannot admit unique solution since $\rho(A) = 2 < 3$ (number of unknowns).

(ii) No solution : The system admits no solution if $k^2 - 5k + 6 \neq 0$, i.e $k \neq 2$ and $k \neq 3$ , so that $\rho(A) \neq \rho(A|B)$.

(iii) Infinite solutions : The system admits infinite solutions if $k = 2$ and $k = 3$, so that $\rho(A) = \rho(A|B) = 2 < 3$ (number of unknowns).

## 4.5 Eigenvalues and Eigenvectors

**Definition 4.19 (Eigenvalue and Eigenvector)** Let $A$ be an $n \times n$ matrix with real entries. A scalar $\lambda \in \mathbb{R}$ or $\mathbb{C}$ is called an eigenvalue of $A$ if there exists a nonzero vector $\bar{v} \in \mathbb{R}^n$ such that $A\bar{v} = \lambda\bar{v}$ and vector $\bar{v}$ is called eigenvector corresponding to the eigenvalue $\lambda$.

**Remarks** :

1. If $\lambda$ is an eigenvalue of $A$, then there exists a vector $\bar{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $A\bar{v} = \lambda\bar{v}$. If $I$ denotes an $n \times n$ identity matrix and $\bar{0}$ denote the zero vector, then we get,
$$A\bar{v} = \lambda I\bar{v} \Rightarrow (\lambda I - A)\bar{v} = \bar{0},$$
a homogeneous system of linear equations in $x_1, \dots, x_n$.
Further, it has a nontrivial solution $\bar{v} \neq 0$ if and only if $\det(\lambda I - A) = 0$.

2. $\det(\lambda I - A)$ is a polynomial in $\lambda$ and we shall denote it by $c(\lambda)$ henceforward, which is called as characteristic polynomial of $A$. Let $c(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0$. The roots $\lambda_1, \lambda_2, \cdots \lambda_n$ (not necessarily distinct) of the equation $c(\lambda) = 0$ are called characteristic roots i.e. eigenvalues of $A$.

**Example 4.32** Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Find eigenvalues of $A$.

Solution : Consider $\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix}$.
Thus $c(\lambda) = det(\lambda I - A) = (\lambda - 1)(\lambda - 3) - 8 = (\lambda - 5)(\lambda + 1)$. Thus $c(\lambda) = 0 \Rightarrow \lambda_1 = 5$ and $\lambda_2 = -1$ are the eigenvalues of $A$.



**Example 4.33** Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$. Find eigenvalues of $A$.

Solution : Consider
$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & 0 & 1 \\ -1 & \lambda - 2 & -1 \\ -2 & -2 & \lambda - 3 \end{bmatrix}.$$

$$\begin{aligned} \text{Thus, } c(\lambda) &= \det(\lambda I - A) \\ &= (\lambda - 1)[(\lambda - 2)(\lambda - 3) - 2] - 0 + 1[2 + 2(\lambda - 2)] \\ &= \lambda^3 - 6\lambda^2 + 11\lambda - 6. \end{aligned}$$

Thus $c(\lambda) = 0 \Rightarrow \lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ are the eigenvalues of $A$.

**Example 4.34** Find eigenvalues of $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$.

Solution : Consider $\lambda I - A =$
$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & 6 & 4 \\ 0 & \lambda - 4 & -2 \\ 0 & 6 & \lambda + 3 \end{bmatrix}.$$

Thus $c(\lambda) = \det(\lambda I - A) = (\lambda - 1)[(\lambda - 4)(\lambda + 3) + 12] + -6(0) + 4(0) = \lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda - 1)^2$. Thus $c(\lambda) = 0 \Rightarrow \lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$ are the eigenvalues of $A$.

**Example 4.35** Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find eigenvalues of $A$.

Solution : Consider $\lambda I - A = \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix}$.
Thus $c(\lambda) = \det(\lambda I - A) = \lambda^3 - 3\lambda - 2$. Thus $c(\lambda) = 0 \Rightarrow \lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = 2$ are the eigenvalues of $A$.

**Example 4.36** Find eigenvalues and eigen-vectors of $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$.
Solution : $c(\lambda) = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$.
Suppose $\bar{v_1} = \begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

Let $(\lambda_1 I - A)\bar{v_1} = \bar{0} \Rightarrow \left( \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 & -4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Writing augmented matrix $\begin{bmatrix} 0 & -4 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ and applying the row operation $R_1 \to R_1 - 4R_2$ we reduce to, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$.
The variable $x$ is free and $y$ leading. So put $x = t \in \mathbb{R}$ and $y = 0$ to get, $\bar{v_1} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Thus an eigenvector corresponding to eigen value 2 is $\bar{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Suppose $\bar{v_2} = \begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$ Let $(\lambda_2 I - A)\bar{v_2} = \bar{0} \Rightarrow \left( \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The augmented matrix is $\begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The variable $y$ is free and $x$ is leading. So put $y = t \in \mathbb{R}$ and $x - 4y = 0 \Rightarrow x = 4t$ so that $\bar{v_2} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Thus an eigenvector corresponding to eigenvalue 3 is $\bar{v_2} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

**Example 4.37** Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.
Solution : $c(\lambda) = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda_1 = 4, \lambda_2 = -1$.

1. Eigenvector $\bar{v_1} = \begin{bmatrix} x \\ y \end{bmatrix}$ corresponding to $\lambda_1 = 4$.

   Let $(\lambda_1 I - A)\bar{v_1} = \bar{0} \Rightarrow \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Writing augmented matrix $\begin{bmatrix} 3 & -2 & 0 \\ -3 & 2 & 0 \end{bmatrix}$ and applying the row operation $R_2 \to R_1 + R_2$, we reduce to $\begin{bmatrix} 3 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The variable $y$ is free and $x$ is leading. So put $y = t \in \mathbb{R}$ and $3x - 2t = 0 \Rightarrow x = \frac{2t}{3}$ to get, $\bar{v_1} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{t}{3} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ Thus $\bar{v_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector.



2. Eigenvector $\bar{v_2} = \begin{bmatrix} x \\ y \end{bmatrix}$ corresponding to $\lambda_2 = -1$

   Let $(\lambda_2 I - A)\bar{v_2} = \bar{0} \Rightarrow \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Writing augmented matrix and applying row operations
   $R_2 \to R_2 + \frac{3}{2}R_1$ and $R_1 \to -\frac{1}{2}R_1$ we reduce to $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
   The variable $y$ is free and $x$ leading. So put $y = t \in \mathbb{R}$ and $x + y = 0 \Rightarrow x = -t$, to get $\bar{v_2} = \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus $\bar{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector.

**Example 4.38** Let $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$.
Find eigenvalues and eigenvectors of $A$.
Solution : $c(\lambda) = \lambda^3 - 4\lambda^2 - \lambda + 4 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 4, \lambda_3 = -1$

1. Eigenvector $\bar{v_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to $\lambda_1 = 1$.

   Let $(\lambda_1 I - A)\bar{v_1} = \bar{0} \Rightarrow \begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
   Writing augmented matrix and applying the row operations
   $R_2 \to -R_2, R_1 \to -\frac{1}{3}R_1, R_1 \to R_1 - R_2,\ R_3 \to R_3 - R_2,$

$R_2 \to R_2 - R_3, R_3 \to \frac{1}{2}R_3$, we reduce to $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

The variable $z$ is free and $x, y$ are leading variables. So put $z = t \in \mathbb{R}$ and $y + z = 0 \Rightarrow y = -t, x = 0$ to get,

$\bar{v_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Thus $\bar{v_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector.

2. Eigenvector $\bar{v_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to $\lambda_2 = 4$

Let $(\lambda_2 I - A)\bar{v_2} = \bar{0} \Rightarrow \begin{bmatrix} 0 & -6 & -6 \\ -1 & 1 & -2 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Writing augmented matrix and applying row operations $R_2 \to R_2 + R_3, R_1 \to -\frac{1}{6}, R_2 \to R_2 - 5R_1$, we reduce to

$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 7 & 0 \end{bmatrix}$. The variable $z$ is free and $x, y$ are leading. So put $z = t \in \mathbb{R}$ and $y + z = 0 \Rightarrow y = -t$, and $x + 4y + 7z = 0 \Rightarrow x = -3t$ to get $\bar{v_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$. Thus an eigenvector corresponding to $\lambda_2 = 4$ is $\bar{v_2} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$.

3. Eigenvector $\bar{v_3} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to $\lambda_3 = -1$.

Let $(\lambda_3 I - A)\bar{v_3} = \bar{0} \Rightarrow \begin{bmatrix} -5 & -6 & -6 \\ -1 & -4 & -2 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.



Writing augmented matrix and applying row operations $R_3 \to R_3 + R_2, -R_1, -R_2, R_1 \to R_1 - 5R_2$, we reduce to

$\begin{bmatrix} 0 & -14 & -4 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The variable $z$ is free and $x, y$ are leading. So put $z = t \in \mathbb{R}$ and $-14y - 4z = 0 \Rightarrow y = -\frac{2}{7}t$, and $x + 4y + 2z = 0 \Rightarrow x = -\frac{6}{7}t$ to get $\bar{v_3} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{t}{7} \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$. Thus an eigenvector corresponding to $\lambda_3 = -1$ is $\bar{v_3} = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$.

**Example 4.39** Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}.$$

Solution : $c(\lambda) = 0 \Rightarrow (\lambda - 3)(\lambda - 2)^2 \Rightarrow \lambda_1 = 3, \lambda_2 = \lambda_3 = 2$ (Note that eigenvalues are repeated).

1. Eigenvector $\bar{v_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to $\lambda_1 = 3$.

Let $(\lambda_1 I - A)\bar{v_1} = \bar{0} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Writing augmented matrix and applying the row operation $R_3 \to R_3 + R_2$, we reduce to $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The variable $z$ is free and $x, y$ are leading variables. So put $z = t \in \mathbb{R}$ and $2y + z = 0 \Rightarrow y = -\frac{t}{2}$ and $x - y = 0 \Rightarrow x = -\frac{t}{2}$ to get,

$\bar{v_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{t}{2} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Thus $\bar{v_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector.

2. Eigenvector $\bar{v_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to $\lambda_2 = 2$. Let

   $(\lambda_2 I - A)\bar{v_2} = \bar{0} \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

   Writing augmented matrix and applying row operations

   $R_3 \to \frac{1}{2}R_3 + R_2$, we reduce to $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   The variable $x$ is free and $y = 0, z = 0$ to get

   $\bar{v_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Thus $\bar{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector.

(Note : Here we get only two eigenvectors for the obtained two eigenvalues. This need **not** be true in general as shown in an example below.)

**Example 4.40** Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}.$$

Solution : $c(\lambda) = 0 \Rightarrow \lambda(\lambda - 1)^2 \Rightarrow \lambda_1 = 0, \lambda_2 = \lambda_3 = 1$. Note that $A$ has repeated eigenvalues.

1. Eigenvector $\bar{v_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to $\lambda_1 = 0$.

   Let $(\lambda_1 I - A)\bar{v_1} = \bar{0} \Rightarrow \begin{bmatrix} -1 & 6 & 4 \\ 0 & -4 & -2 \\ 0 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

   Writing augmented matrix and applying the row operation

   $\frac{1}{2}R_3, R_3 \to R_3 - R_2$,



   we reduce to $\begin{bmatrix} -1 & 6 & 4 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The variable $z$ is free and $x, y$ are leading variables. So put $z = t \in \mathbb{R}$ and $2y + z = 0 \Rightarrow y = -\frac{t}{2}$ and $-x + 6y + 4z = 0 \Rightarrow x = t$ to get, $\bar{v_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Thus

   $\bar{v_1} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ is an eigenvector.

2. Eigenvector $\bar{v_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to $\lambda_2 = \lambda_2 = 1$.

   Let $(\lambda_2 I - A)\bar{v_2} = \bar{0} \Rightarrow \begin{bmatrix} 0 & 6 & 4 \\ 0 & -3 & -2 \\ 0 & 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

   Writing augmented matrix and applying row operations $\frac{1}{2}R_1$, $\frac{1}{2}R_3, -R_2, R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$, we reduce to

   $$\begin{bmatrix} 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   The variables $x$ and $z$ are free and $y$ is leading variable. So put $x = t \in \mathbb{R}, z = s \in \mathbb{R}$ and $3y + 2z = 0 \Rightarrow y = -\frac{2s}{3}$, to get

   $\bar{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1t + 0s \\ 0t - \frac{2s}{3} \\ 0t + 1s \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{s}{3} \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$. Thus $\bar{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and

   $\bar{v_3} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$ are eigenvectors.

**Theorem 4.1 (Cayley Hamilton Theorem)** Every square matrix satisfies its characteristic polynomial.

**Example 4.41** Verify Cayley Hamilton theorem and use it to find $A^{-1}$, if it exists. $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

Solution : Here $c(\lambda) = \lambda^3 - 20\lambda + 8$. We check whether $c(A) = \mathbf{O}$, where $\mathbf{O}$ denotes the $n \times n$ zero matrix. We have

$$A^2 = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}, A^3 = \begin{bmatrix} 12 & 20 & 60 \\ 20 & 55 & -60 \\ -40 & -80 & -88 \end{bmatrix}. \text{ Hence}$$

$$\begin{aligned} c(A) &= A^3 - 20A + 8I \\ &= \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - 20 \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus $c(A) = A^3 - 20A + 8I = \mathbf{O}$, so that Cayley Hamilton theorem is verified. Further $\det(A) = -8 \neq 0$, hence $A^{-1}$ exists.

Since $A^3 - 20A + 8I = \mathbf{O}$, post-multiplying by $A^{-1}$ we get,

$A^3A^{-1} - 20AA^{-1} + 8IA^{-1} = \mathbf{O} \Rightarrow A^2 - 20I + 8A^{-1} = \mathbf{O}$

$\Rightarrow A^{-1} = \frac{1}{8}(20I - A^2)$

$$\Rightarrow A^{-1} = \frac{1}{8} \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} - \begin{bmatrix} -4 & -4 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

## 4.6 Exercises

1. Why are the following matrices not in row echelon form ?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

2. Why are the matrices given below not in reduced row echelon form?



$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

3. Determine whether the matrices given below are in row echelon form or reduced row echelon form?
$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -7 & 5 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix}.$$

4. Apply elementary row and elementary column transformations to reduce matrix $A = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 5 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
   (Hint : Use $R_2 - R_1, R_3 - 2R_1, C_2 - 4C_1, C_3 - 3C_1, C_4 - 2C_1, -\frac{1}{2}C_2, C_2 - C_3, C_4 - 2C_2, C_4 - C_3$).

5. Find the rank of the following matrices :
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 4 & 4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}.$$
   (Ans : 3, 1, 3, 2)

6. Solve the following systems using Gauss elimination method:
   (a) $x + y + 2z = 8, -x - 2y + 3z = 1, 3x - 7y + 4z = 10$.
   (b) $2x + 2y + 2z = 0, -2x + 5y + 2z = 1, 8x + y + 4z = -1$.
   (c) $-2y + 3z = 1, 3x + 6y - 3z = -2, 6x + 6y + 3z = 5$.
   (d) $3x + 2y - z = -15, \quad 5x + 3y + 2z = 0, \quad 3x + y + 3z = 11, -6x - 4y + 2z = 30$.
   (e) $2x + y + 5z + w = 5, x + y - 3z - 4w = -1, 3x + 6y - 2z + w = 8, 2x + 2y + 2z - 3w = 2$.

   (Ans (a): $x = 3, y = 1, z = 2$, (b) $x = -\dfrac{1+3t}{7}, y = \dfrac{1-4t}{7}, z = t$,
   (c) Inconsistent system (d) $x = -4, y = 2, z = 7$
   (e) $x = 2, y = \dfrac{1}{5}, z = 0, w = \dfrac{4}{5}$.)

7. Solve the following systems using Gauss Jordan method :

   (a) $4x - 8y = 12$, $3x - 6y = 9$, $-2x + 4y = -6$.

   (b) $2x - y - 3z = 0$, $-x + 2y - 3z = 0$, $x + y + 4z = 0$.

   (c) $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$.

   (d) $2x + z = 4$, $x - 2y + 2z = 7$, $3x + 2y = 1$.

   (e) $2x + y + 2z + w = 6$, $6x - 6y + 6z + 12w = 36$, $4x + 3y + 3z - 3w = -1$, $2x + 2y - z + w = 10$.

   (Ans (a) $x = 3 + 2t, y = t$, (b) Trivial solution i.e. $x = y = z = 0$ (c) $x = 1, y = 2, z = -1$ (d) $x = \dfrac{4-t}{2}, y = \dfrac{-5+3t}{4}, z = t$, (e) $x = 2, y = 1, z = -1, w = 3$)

8. Show that the following systems are inconsistent:

   (a) $x + y + z = 3$, $2x - y + 3z = 2$, $3x - 2y + z = 4$, $4x + y + 5z = 2$.

   (b) $x + y - 2z + 3w = 4$, $2x + 3y + 3z - w = 3$, $5x + 7y + 4z + w = 5$.

9. Show that the following systems have infinite solutions :

   (a) $2x + z = 4$, $x - 2y + 2z = 7$, $3x + 2y = 1$.

   (b) $2x - y + 3z = 1$, $3x + 2y + z = 3$, $x - 4y + 5z = -1$.

   (c) $x+y-2z+4w = 5, 2x+2y-3z+w = 3, 3x+3y-4z-2w = 1$.

   (d) $2x - 5y + 3z - 4w + 2u = 4$, $3x - 7y + 2z - 5w + 4u = 9$, $5x - 10y - 5z - 4w + 7u = 22$.

   (Hint : Check that $\rho(A) = \rho(A|B) < n$(number of variables)).

10. Find all possible solutions of the systems given below (if they exist):

   (a) $x + y + z = 4$, $2x + 5y - 2z = 3$.



   (b) $2x+y+3z+6w = 0$, $3x-y+z+3w = 0$, $-x-2y+3z = 0$, $-x - 4y - 2z - 7w = 0$.

   (c) $2x - y - z = 2$, $x + 2y + z = 2$, $4x - 7y - 5z = 2$.

   (d) $2x - 3y + 7z = 5$, $3x + y - 3z = 13$, $2x + 19y - 47z = 32$.

   (e) $x - 3y + 2z - w + 2u = 2$, $3x - 9y + 7z - w + 3u = 7$, $2x - 6y + 7z + 4w - 5u = 7$.

   Solutions :

   (a) $x = \frac{17-7t}{3}$, $y = \frac{-5+4t}{3}$, $z = t, t \in \mathbb{R}$.

   (b) $x = -t, y = -t, z = -t, w = t, t \in \mathbb{R}$.

   (c) $x = \frac{t+6}{2}$, $y = \frac{2-3t}{5}$, $z = t, t \in \mathbb{R}$.

   (d) System is inconsistent hence no solution exists, .

   (e) $y = t$, $w = s$, $u = r$ $(t, s, r \in \mathbb{R})$ are free variables, $x = 3t + 5s - 8r$, $z = 1 - 2s + 3r$.

11. Check whether the systems given below have unique solution? If not, find all possible solutions.

   (a) $x + 2y + z = 3$, $2x + 5y - z = -4$, $3x - 2y - z = 5$.

   (b) $x + 2y + 2z = 1$, $2x + 2y + 3z = 3$, $x - y + 3z = 5$.

   (c) $x + y + z = 6, x - y + 2z = 5, 3x + y + z = 8$, $2x - 2y + 3z = 7$.

   (d) $2x - y + 3z = 1$, $3x + 2y + z = 3$, $x - 4y + 5z = -1$.

   (e) $2x + y + 2z + w = 6$, $5x - 6y + 6z + 12w = 36$, $4x + 3y + 3z - 3w = -1$, $2x + 2y - z + w = 10$.

   (f) $2x+y-5z+w = 8$, $x+3y-6w = -15$, $2y-z+2w = -5$, $x + 4y - 7z + 6w = 0$.

   (g) $4x-y+2z+w = 0$, $2x+3y-z-2w = 0$, $7x-4z-5w = 0$, $2x - 11y + 7z + 8w = 0$.

   Solutions :

(a) Unique solution : Since $\rho(A) = \rho(A|B) = 3$ (3 unknowns) and $x = 2, y = -1, z = 3$.

(b) Unique solution : Since $\rho(A) = \rho(A|B) = 3$ (3 unknowns) and $x = 1, y = -1, z = 1$.

(c) Unique solution : Since $\rho(A) = \rho(A|B) = 3$ (3 unknowns) and $x = 1, y = 2, z = 3$.

(d) Infinite solutions : $x = \frac{5-7t}{7}$, $y = \frac{3+7t}{7}$, $z = t, t \in \mathbb{R}$.

(e) Unique solution : Since $\rho(A) = \rho(A|B) = 4$ (4 unknowns) and $x = 2, y = 1, z = -1, w = 3$.

(f) Unique solution : Since $\rho(A) = \rho(A|B) = 4$ (4 unknowns) and $x = 3, y = -4, z = -1, w = 1$.

(g) Infinite solutions : $x = \frac{-5t-8}{14}$, $y = \frac{4t+5s}{7}$, $z = t$, $w = s$, $t \in \mathbb{R}$.

12. Find the values of $\lambda$ for which the following systems admit a unique solution :

(a) $2x + 3y + 5z = 9, 7x + 3y - 2z = 8, 2x + 3y + \lambda z = 5$.

(b) $x + y + z = 6, x + 2y + 3z = 10, x + 2y + 4z = \lambda$.
(Ans (a) : $\lambda \neq 5$ , (b) : $\lambda = 11$)

13. Find values of $\lambda$ for which the following systems admits no solution:

(a) $2x + y = 2, x - z = 3, y + 2z = \lambda$.

(b) $x + y + z = 6, x + 2y + 3z = 10, x + 2y + 3z = \lambda$.

(c) $2x-3y+6z-5w = 3, y-4z+w = 1, 4x-5y+8z-9w = \lambda$.
((Ans (a) : $\lambda \neq -4$ (b) $\lambda \neq 10$) (c) $\lambda \neq 7$)

14. Find the values of $\lambda$ for which the following systems infinite solutions:

(a) $2x + y = 3, x - z = \lambda, y + 2z = 1$.

(b) $x + y + z = 6, x + 2y + 3z = 10, x + 2y + 3z = \lambda$.
((Ans (a): $\lambda = 1$, (b) $\lambda = 10$)



15. Find the value of $\lambda$ and $\mu$ so that the systems given below admit (a) Unique solution (b) No solution (c) infinite number of solutions.

(a) $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda = \mu$ has
(Ans : (a) $\lambda \neq 3$, $\mu \in \mathbb{R}$, (b) $\lambda = 3$, $\mu \neq 10$, (c) $\lambda = 3, \mu = 10$).

(b) $2x + 3y + 5z = 9, 7x + 3y - 2z = 8, 2x + 3y + \lambda z = \mu$.
(Ans : (a) $\lambda \neq 5$, $\mu \in \mathbb{R}$, (b) $\lambda = 5, \mu \neq 9$, (c) $\lambda = 5, \mu = 9$).

16. Find the eigenvalues of the following matrices :

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(Ans(a) : $(4, -1)$, (2,2) (2,4) (b) : 1,2,3)

17. Find eigenvalues and eigenvectors of the following matrices:

$$\begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Solutions :

(a) $\lambda_1 = 4$ , $\bar{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_2 = 9$, $\bar{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\lambda_1 = 5$, $\bar{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_2 = -1$, $\bar{v_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(c) $\lambda_i = 1, -1, 3$, $\bar{v_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\bar{v_2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\bar{v_3} = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

(d) $\lambda_i = 3, 2, 5$, $\bar{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\bar{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\bar{v_3} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(e) $\lambda_i = 1, 2, 3$, $\bar{v_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\bar{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\bar{v_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

18. Find eigenvalues and eigenvectors of the following matrices:

$$\begin{bmatrix}1&2&2\\0&2&1\\-1&2&2\end{bmatrix},\ \begin{bmatrix}1&2&3\\0&2&3\\0&0&2\end{bmatrix},\ \begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix},\ \begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix}.$$

Solutions

(a) $\lambda_i = 1, 2, 2,\ \bar{v_1} = \begin{bmatrix}1\\0\\0\end{bmatrix},\ \bar{v_2} = \begin{bmatrix}2\\1\\0\end{bmatrix}$

(b) $\lambda_i = 1, 2, 2,\ \bar{v_1} = \begin{bmatrix}1\\1\\-1\end{bmatrix},\ \bar{v_2} = \begin{bmatrix}2\\1\\0\end{bmatrix}$

(c) $\lambda_i = 2, 2, 2,\ \bar{v_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ (d) $\lambda_i = 1, 1, 1,\ \bar{v_1} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

19. Find eigenvalues and eigenvectors of the following matrices:

$$\begin{bmatrix}-3&-7&-5\\2&4&3\\1&2&2\end{bmatrix}\ \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix},\ \begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix},\ \begin{bmatrix}1&0&0\\2&0&1\\3&1&0\end{bmatrix}.$$

(a) $\lambda_i = 1, 1, 1,\ \bar{v_1} = \begin{bmatrix}-3\\1\\1\end{bmatrix}$

(b) $\lambda_i = 1, 1, 1,\ \bar{v_1} = \begin{bmatrix}1\\0\\0\end{bmatrix},\ \bar{v_2} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

(c) $\lambda_i = 3, 1, 1,\ \bar{v_1} = \begin{bmatrix}1\\1\\0\end{bmatrix},\ \bar{v_2} = \begin{bmatrix}1\\-1\\0\end{bmatrix},\ \bar{v_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

(d) $\lambda_i = -1, 1, 1,\ \bar{v_1} = \begin{bmatrix}0\\-1\\1\end{bmatrix},\ \bar{v_2} = \begin{bmatrix}0\\1\\1\end{bmatrix},\ \bar{v_3} = \begin{bmatrix}1\\0\\0\end{bmatrix}$



20. Verify Cayley Hamilton theorem and hence find $A^{-1}$ for the following matrices :

(a) $A = \begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$ (b) $A = \begin{bmatrix}1&2&2\\0&2&1\\-1&2&2\end{bmatrix}$

Ans (a): $c(\lambda) = \lambda^3 - 3\lambda^2 - 8\lambda + 2$ and $A^{-1} = \frac{1}{2}\begin{bmatrix}1&-1&1\\-8&6&-2\\5&-3&1\end{bmatrix}$

(b) $c(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$ and $A^{-1} = \frac{1}{4}\begin{bmatrix}2&0&-2\\-1&4&-1\\2&-4&2\end{bmatrix}$.

# Chapter 5

# Analytical Geometry of Two Dimensions

## 5.1 Change of Axes

### 5.1.1 Translation of Axes

Let $OX$ and $OY$ be the original rectangular frame of reference of axes. Let $O'X'$ and $O'Y'$ be the new axes parallel to the original axes. Let $O'(h,k)$ be the new origin. Let $P$ be any point in the plane with co-ordinates $(x,y)$ and $(x',y')$ with respect to original and new co-ordinate axes respectively.

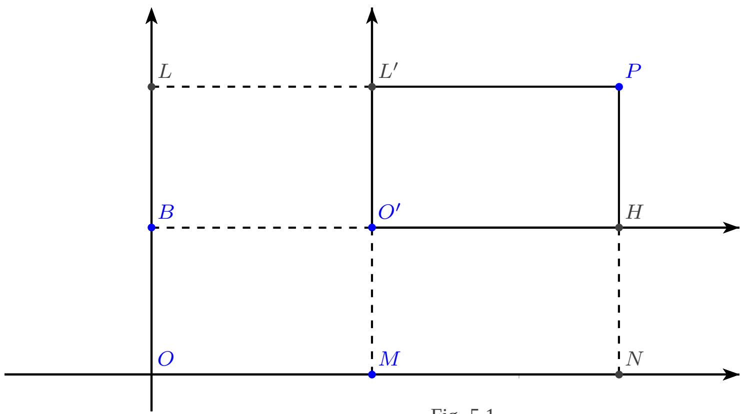



Fig. 5.1

From the new origin $O'$, the perpendicular $O'M$ is drawn on $OX$. Also from the point $P$, the perpendicular $PN$ is drawn to $OX$. Then perpendicular $PN$ meets $O'X'$ in the point $N'$.
From fig. 5.1, we get $x = ON, y = PN, x' = O'N', y' = PN'$. Hence, $x = ON = OM + MN = OM + O'N' = h + x'$. Similarly $y = PN = PN' + N'N = PN' + O'M = y' + k$. Thus

$$x = x' + h, \qquad y = y' + k \tag{1}$$





Equations (1) are the relations between the co-ordinates of the same point $P$ referred to the two frames of reference of axes. These equations are called the equations of translation.

**Example 5.1** The origin is shifted to the point $(h,2)$, find the value of $h$ so that the transformed equation of locus given by the equation $x^2+4x+3y = 5$ will not contain the first degree term in $x$.
**Solution:** We have $x^2 + 4x + 3y - 5 = 0$ (1)
Since origin is shifted to the point $(h,k) = (h,2)$. We know equations of translations

$$x = x' + h \qquad y = y' + k = y' + 2 \tag{2}$$

Using (2) in (1), we get $(x'+h)^2 + 4(x'+h) + 3(y'+2) - 5 = 0$. Hence, $x'^2 + (2h+4)x' + 3y' + h^2 + 4h + 1 = 0$. This equation will not contains first degree term in $x'$ if $2h + 4 = 0$. Therefore $h = -2$.

**Rotation of Axes :**

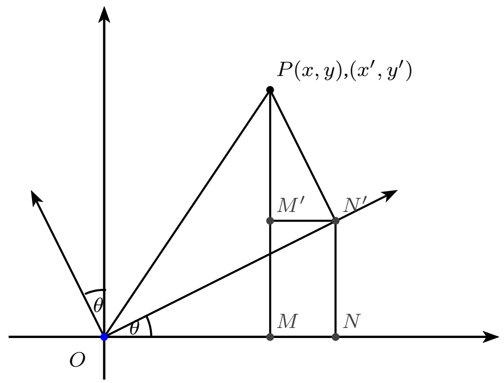



Fig. 5.2

Let $OX$ and $OY$ be the original rectangular frame of reference of axes. Let $OX'$ and $OY'$ be the new positions of axes obtained by rotating the original rectangular axes $OX$ and $OY$ through an angle $\theta$ keeping origin fixed. From point $P$, the perpendicular $PM$ is drawn on $OX$. Also the perpendicular $PN'$ is drawn on $OX'$. Then $N'N$ is drawn perpendicular to $OX$ and $N'M'$ is drawn perpendicular $PM$. Then $\angle M'PN' = \theta$. We have $x = PM, y = PM, x' = ON', y' = PN'$. From $\triangle ONN'$, we get $\cos\theta = \frac{ON}{x'}, \sin\theta = \frac{NN'}{x'}$. Thus,

$$ON = x'\cos\theta, \quad NN' = x'\sin\theta \qquad (1)$$

From $\triangle PM'N'$, $\cos\theta = \frac{PM'}{y'}, \sin\theta = \frac{M'N'}{y'}$

$$PM' = y'\cos\theta, \quad M'N' = y'\sin\theta \qquad (2)$$

$$\begin{aligned} \text{Now } x &= OM = ON - MN \\ &= ON - M'N' \qquad (\because MN = M'N') \\ &= x'\cos\theta - y'\sin\theta. \\ \text{Similarly, } \quad y &= PM = PM' + M'M \\ &= PM' + N'N \qquad (\because M'M = N'N) \\ &= y'\cos\theta + x'\sin\theta. \end{aligned}$$

$$\begin{aligned} \text{Thus,} \qquad x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \qquad (2) \end{aligned}$$

The equations (2) are called the equations of rotations.

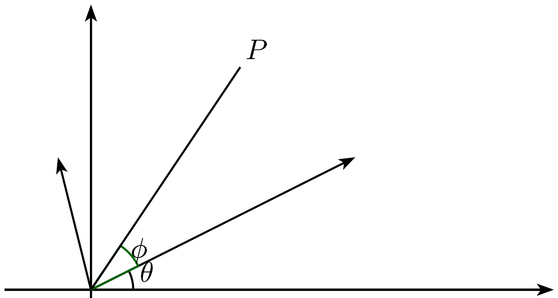



Fig. 5.3

**Another method.** Let $OX$ and $OY$ be the original rectangular frame of reference of axes. Let $OX'$ and $OY'$ be the new positions of axes obtained



by rotating the original rectangular axes $OX$ and $OY$ through an angle $\theta$ keeping origin fixed. Let $(x, y)$ be the coordinates of the point $P$ w.r.t. the $OXY$ coordinate system and $(x, y)$ be the coordinates of the point $P$ w.r.t. the $OX'Y'$ coordinate system. If $OP$ makes an angle $\phi$ w.r.t. the positive direction of $x'$ axis i.e w.r.t. $OX'$ then $OP$ makes an angle $(\phi+\theta)$ w.r.t. the positive direction of $x$ axis i.e w.r.t. $OX$. Hence, $x' = r\cos\phi, y' = r\sin\phi$ and $x = r\cos(\phi+\theta), y = r\sin(\phi+\theta)$.

$$\begin{aligned} \text{Thus,} \quad x &= r\cos(\phi+\theta) = r(\cos\phi\cos\theta - \sin\phi\sin\theta) \\ &= x'\cos\theta - y'\sin\theta \qquad (3) \\ y &= r\sin(\phi+\theta) = r(\sin\phi\cos\theta + \cos\phi\sin\theta) \\ &= x'\sin\theta + y'\cos\theta \qquad (4) \end{aligned}$$

## 5.2 Removal of $xy$ term

To determine the angle $\theta$ through which the axes should be rotated so that the transformed form of the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (5)$$

is free from product term. Suppose axes are rotated through an angle $\theta$. Then the equations of rotation are

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \qquad (2) \end{aligned}$$

Using equations (2) in equation (5) we get
$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$
$+ b(x'\sin\theta + y'\cos\theta)^2 + 2g(x'\cos\theta - y'\sin\theta) + 2f(x'\sin\theta + y'\cos\theta)$
$+ c = 0.$

$(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)x'^2 + 2(-a\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta + b\sin\theta\cos\theta)x'y' + (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)y'^2 + 2(g\cos\theta + f\sin\theta)x' + 2(-g\sin\theta + f\cos\theta)y' + c = 0$ i. e.

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c = 0 \qquad (3)$$

where

$$\begin{aligned} a' &= a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta \\ h' &= -a\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) + b\sin\theta\cos\theta \\ b' &= a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta \\ g' &= g\cos\theta + f\sin\theta, \quad f' = -g\sin\theta + f\cos\theta. \end{aligned}$$

The transformed form of equation (1) is equation (3). It will not contain product term $x'y'$ if $h' = 0$ i. e. if

$$-a\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) + b\sin\theta\cos\theta = 0.$$

Multiplying by 2

$$\begin{aligned} 2h(\cos^2\theta - \sin^2\theta) &= (a-b)2\sin\theta\cos\theta \\ 2h\cos 2\theta &= (a-b)\sin 2\theta \\ \tan 2\theta &= \frac{2h}{a-b} \text{ if } a-b \neq 0 \\ \theta &= \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right) \end{aligned}$$

On the other hand, if $a = b$ then axes are rotated through an angle $\theta = \frac{\pi}{4}$, so that $h' = 0$. Thus, if if $a = b$ then axes are rotated through an angle $\theta = \frac{\pi}{4}$, and if $a \neq b$ axes are rotated through an angle $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$, then the transformed form of equation (1) will not contains the product term $xy$.

**Example 5.2** Shift the origin to a suitable point so that the equation $x^2 - 6x - 4y - 1 = 0$ will be in the form $x^2 = 4by$ state the value of $b$.

**Solution:** We have

$$x^2 - 6x - 4y - 1 = 0 \qquad (1)$$

Shift the origin to point $(h, k)$. The equations of translations are

$$x = x' + h, y = y' + k \qquad (2)$$



Using (2) in (1)

$$\begin{aligned} (x'+h)^2 - 6(x'+h) - 4(y'+k) - 1 &= 0 \\ x'^2 + 2hx' + h^2 - 6x' - 6h - 4y' - 4k - 1 &= 0 \end{aligned}$$

$$x'^2 = 4y' + (6-2h)x' + 4k + 1 - h^2 \qquad (3)$$

Equation (3) will becomes $x^2 = 4by$ if $6 - 2h = 0$, and $4k + 1 - h^2 = 0$. Hence, $h = 3$, and $4k = 8$ i.e. $k = 2$. Thus $(h, k) = (3, 2)$ and $b = 1$.

**Example 5.3** Change the origin to point $(\alpha, \beta)$ and transform the equation $x^2 - 2xy + 3y^2 - 10x + 22y + 30 = 0$. Find $(\alpha, \beta)$ if the transformed equation does not contain the first degree terms in the new co-ordinates.

**Solution:** We have

$$x^2 - 2xy + 3y^2 - 10x + 22y + 30 = 0 \qquad (1)$$

Let $(h, k) = (\alpha, \beta)$. Shift the origin to the point $(\alpha, \beta)$.

$$x = x' + \alpha, y = y' + \beta \qquad (2)$$

Using (2) in (1), we get
$(x'+\alpha)^2 - 2(x'+\alpha)(y'+\beta) + 3(y'+\beta)^2 - 10(x'+\alpha) + 22(y'+\beta) + 3 = 0$
$x'^2 - 2x'y' + y'^2 + (2\alpha - 2\beta - 10)x' + (-2\alpha + 6\beta + 22)y'$
$+ (\alpha^2 - 2\alpha\beta + 3\beta^2 - 10\alpha + 22\beta + 3) = 0 \qquad (3)$
Since the transformed equation (3) does not contains first degree terms in $x'$ and $y'$. Therefore $2\alpha - 2\beta - 10 = 0$ and $-2\alpha + 6\beta + 22 = 0$. Thus,

$$\begin{aligned} \alpha - \beta - 5 &= 0 \qquad (4) \\ -\alpha + 3\beta + 11 &= 0 \qquad (5) \end{aligned}$$

Solving equations (4) and (5) we get $\alpha = 2, \beta = -3$. Thus $(\alpha, \beta) = (2, -3)$.

**Invariants:-**
The quantities which remain unchanged by change of axes are called invariants. Translation of axes or rotation of axes or both are called change of axes.

**Theorem 5.1** If by rotating the axes through an angle $\theta$, without changing the origin the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ is transformed into $a'x'^2+2h'x'y'+by'^2+2g'x'+2f'y'+c=0$ then
(i) $a+b=a'+b'$ and (ii) $ab-h^2=a'b'-h'^2$.

**Proof:** The given equation is $ax^2+2hxy+by^2+2gx+2fy+c=0$ (1)
Suppose axes are rotated through an angle $\theta$. Then the equations of rotation are

$$x = x'\cos\theta - y'\sin\theta, \qquad y = x'\sin\theta + y'\cos\theta \qquad (2)$$

Using (2) in equation (1)

$$\begin{aligned}
& a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)\\
+ \quad & b(x'\sin\theta + y'\cos\theta)^2 + 2g(x'\cos\theta - y'\sin\theta)\\
+ \quad & 2f(x'\sin\theta + y'\cos\theta) + c = 0\\
& a(\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)x'^2\\
+ \quad & (-2a\sin\theta\cos\theta + 2h\cos^2\theta - 2h\sin^2\theta + 2b\sin\theta\cos\theta)x'y'\\
+ \quad & (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)y'^2\\
+ \quad & (2g\cos\theta + 2f\sin\theta)x' + (-2g\sin\theta + 2f\cos\theta)y' + c = 0.
\end{aligned}$$

We may rewrite this equation as

$$a'x^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c = 0, \text{ where}$$

$$\begin{aligned}
a' &= a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta \qquad (3)\\
h' &= -a\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta) + b\sin\theta\cos\theta \qquad (4)\\
b' &= a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta \qquad (5)\\
g' &= g\cos\theta + f\sin\theta, \quad f' = -g\sin\theta + f\cos\theta
\end{aligned}$$

Adding equations (3) and (5), it is easy to see that $a'+b'=a+b$.
We know the formulae,

$$\begin{aligned}
\sin 2\theta = 2\sin\theta\cos\theta, &\qquad \cos 2\theta = \cos^2\theta - \sin^2\theta\\
1+\cos 2\theta = 2\cos^2\theta, &\qquad 1-\cos 2\theta = 2\sin^2\theta \qquad (6)
\end{aligned}$$



Multiply equations (3), (4) and (5) by 2.

$$\begin{aligned}
2a' &= 2a\cos^2\theta + 2h(2\sin\theta\cos\theta) + 2b\sin^2\theta\\
&= a(1+\cos 2\theta) + 2h\sin 2\theta + b(1-\cos 2\theta)\\
&= (a+b) + [(a-b)\cos 2\theta + 2h\sin 2\theta]\\
2h' &= -2a\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta) + 2b\sin\theta\cos\theta\\
&= 2h\cos 2\theta - (a-b)\sin 2\theta\\
2b' &= 2a\sin^2\theta - 2h(2\sin\theta\cos\theta) + 2b\cos^2\theta\\
&= a(1-\cos 2\theta) - 2h\sin 2\theta + b(1+\cos 2\theta)\\
&= a+b-[(a-b)\cos 2\theta + 2h\sin 2\theta]
\end{aligned}$$

Now

$$\begin{aligned}
(2a')(2b') - (2h')^2 &= \{(a+b) + [(a-b)\cos 2\theta + 2h\sin 2\theta]\}\\
&\quad \cdot\{(a+b) - [(a-b)\cos 2\theta + 2h\sin 2\theta]\}\\
&\quad -[2h\cos 2\theta - (a-b)\sin 2\theta]^2\\
&= (a+b)^2 - [(a-b)\cos 2\theta + 2h\sin 2\theta]^2\\
&\quad -[2h\cos 2\theta - (a-b)\sin 2\theta]^2\\
\therefore 4(a'b' - h'^2) &= (a+b)^2 - [(a-b)^2\cos^2 2\theta\\
&\quad +4h(a-b)\sin 2\theta\cos 2\theta + 4h^2\sin^2 2\theta]\\
&\quad -[4h^2\cos^2 2\theta - 4h(a-b)\cos 2\theta\sin 2\theta\\
&\quad +(a-b)^2\sin^2 2\theta]\\
&= (a+b)^2 - (a-b)^2[\cos^2 2\theta + \sin^2 2\theta]\\
&\quad -4h^2[\sin^2 2\theta + \cos^2 2\theta]\\
&= (a+b)^2 - (a-b)^2 - 4h^2 = 4(ab-h^2)\\
a'b' - h'^2 &= ab - h^2
\end{aligned}$$

$\therefore a+b$ and $ab-h^2$ are invariants.

**Example 5.4** By rotating the axes, origin being unchanged the expression $ux+vy$ becomes $u'x'+v'y'$, show that $u^2+v^2=u'^2+v'^2$.

**Solution:** Suppose axes are rotated through an angle $\theta$. Then

$$x = x'\cos\theta - y'\sin\theta, y = x'\sin\theta + y'\cos\theta.$$

Now the expression

$$\begin{aligned}
ux + vy &= u(x'\cos\theta - y'\sin\theta) + v(x'\sin\theta + y'\cos\theta)\\
&= (u\cos\theta + v\sin\theta)x' + (-u\sin\theta + v\cos\theta)y'\\
&= u'x' + v'y'\\
\text{where } u' &= u\cos\theta + v\sin\theta,\ \ v' = -u\sin\theta + v\cos\theta\\
u'^2 + v'^2 &= (u\cos\theta + v\sin\theta)^2 + (-u\sin\theta + v\cos\theta)^2\\
&= u^2\cos^2\theta + 2uv\sin\theta\cos\theta + v^2\sin^2\theta\\
&\quad + u^2\sin^2\theta - 2uv\sin\theta\cos\theta + v^2\cos^2\theta\\
&= u^2 + v^2.
\end{aligned}$$

Therefore $u^2 + v^2$ is invariant.

## 5.3 General Equation of second degree in $x$ and $y$ :

**Theorem 5.2** A general equation of second degree represents a conic.

**Proof:** Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1)
be the general equation of second degree in $x$ and $y$.
We know that the equation (1) represents a pair of lines iff
$\triangle \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ so suppose $\triangle \neq 0$.
Rotate the co-ordinate axes through an angle $\theta = \frac{\pi}{4}$ if $a = b$ and if $a \neq b$ then

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right). \qquad (2)$$

Then the transformed form of equation (1) does not contain the product term in $xy$. Suppose equation (1) is transformed to

$$Ax'^2 + By'^2 + 2Gx' + 2Fy' + C = 0 \qquad (3)$$

**Case I :** Suppose $A = B(\neq 0)$. Then equation (3) represents circle.
**Case II :** Suppose $A \neq B$ and $A \neq 0, B \neq 0$. Then equation (3) can be written as



$$\begin{aligned}
Ax'^2 + 2A\frac{G}{A}x' + By'^2 + 2B\frac{F}{B}y' &= -c\\
A\left(x' + \frac{G}{A}\right)^2 + B\left(y' + \frac{F}{B}\right)^2 &= \frac{G^2}{A} + \frac{F^2}{B} - c = K\\
\text{where } K = \frac{G^2}{A} + \frac{F^2}{B} - c & \qquad (4)
\end{aligned}$$

Now we shift the origin to the point $\left(\frac{-G}{A}, \frac{-F}{B}\right)$. Then equation (4) becomes

$$Ax''^2 + By''^2 = K \qquad (5)$$

If $K = 0$, then equation (5) is homogeneous equation of second degree in $x''$ and $y''$. Equation (5) can be factorised into two linear factors which represents pair of straight lines. If $A$ and $B$ are of same sign then it represents imaginary lines and if $A$ and $B$ are of opposite signs then it represents real lines. If $K \neq 0$. Then dividing equation (5) by $K$

$$\frac{x''^2}{K/A} + \frac{y''^2}{K/B} = 1 \qquad (6)$$

If $K/A$ and $K/B$ are positive then equation (6) represents ellipse. If $K/A$ and $K/B$ are negative then equation (6) represents imaginary ellipse. If $K/A$ and $K/B$ are of opposite signs, then equation (6) represents hyperbola.

**Case III :** Suppose $A = 0$ but $B \neq 0$. Then equation (3) becomes

$$\begin{aligned}
By'^2 + 2Gx' + 2Fy' + c &= 0\\
B\left(y'^2 + 2\frac{F}{B}y'\right) &= -2Gx' - c\\
B\left(y'^2 + 2\frac{F}{B}y' + \frac{F^2}{B^2}\right) &= -2Gx' - c + \frac{F^2}{B}\\
B\left(y' + \frac{F}{B}\right)^2 &= -2G\left(x' - \frac{F^2}{2BG} + \frac{C}{2G}\right).
\end{aligned}$$

Shift the origin to the point $\left(\frac{F^2}{2BG} - \frac{C}{2G}, -\frac{F}{B}\right)$. Then above equation reduces to

$$By''^2 = -2Gx'' \quad \text{i. e.} \quad y''^2 = -\frac{2G}{B}x''.$$

This is the equation of parabola. If $G = 0$, then it represents pair of parallel line and in this case $\triangle = 0$.

**Case IV :** Suppose $B = 0$ but $A \neq 0$. Then equation (3) is transformed to

$$x''^2 = -\frac{2F}{A}y''$$

This is equation of parabola.
Thus every general equation of second degree represents conic.

**Centre of the Conic:**
**Definition:** A point in the plane of conic which bisect every chord of the conic passing through it is called centre of the conic. A conic having centre is called central conic. Circle and ellipse are central conic. Parabola is a non-central conic.
**Proposition:** If the centre of the conic is origin then the equation of the conic is free from linear factor.

**Proof:** Suppose origin is the centre of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Let $y = mx$ (2) be the equation of line passing through origin.
At the point of intersection of line (2) and conic (1)

$$\begin{aligned} ax^2 + 2hx(mx) + bm^2x^2 + 2gx + 2fmx + c &= 0 \\ (a + 2hm + bm^2)x^2 + (2g + 2fm)x + c &= 0 \qquad (3) \end{aligned}$$

This is quadratic in $x$. Therefore it has two roots say $x_1$ and $x_2$.
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two end points of chord $PQ$. But origin $O(0, 0)$ is centre of the conic. Therefore, by definition of centre of conic

$$\frac{x_1 + x_2}{2} = 0, \ \frac{y_1 + y_2}{2} = 0, \ x_1 + x_2 = 0.$$



i.e. sum of roots of (3) $= 0$. By (3)

$$\frac{-(2g + 2fm)}{(a + 2hm + bm^2)} = 0 \quad \text{i.e.} \quad g + fm = 0 \quad \text{for all values of } m.$$

$\Rightarrow g = 0, f = 0$. Thus, coefficient of $x = 0 =$ coefficient of $y$.
Putting $g = 0, f = 0$ in equation (1) we get $ax^2 + 2hxy + by^2 + c = 0$.
This equation is free from first degree terms.

**The Centre of the Conic:**
Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1)
be the equation of the central conic.
Let $(x_1, y_1)$ be the centre of the conic (1). Now we shift the origin to the point $(x_1, y_1)$. Then the equation of translations are

$$x = x' + x_1, \ \ y = y' + y_1 \qquad (2)$$

Using (2) in equation (1)
$a(x' + x_1)^2 + 2h(x'x_1)(y' + x_1) + b(y' + y_1)^2 + 2g(x' + x_1)$
$+ 2f(y' + y_1) + c = 0$
$ax'^2 + 2hx'y' + by'^2 + (ax_1 + hy_1 + g)x_1 + 2(x_1 + by_1 + f)y'$
$+ ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_+1 + c = 0$ (3)
Since origin is the centre of the conic (3), we get coefficient of $x' = 0$ and coefficient of $y' = 0$

$$\begin{aligned} ax_1 + hy_1 + g &= 0 \qquad (4) \\ hx_1 + by_1 + f &= 0 \qquad (5) \end{aligned}$$

Solving equations (4) and (5) we get

$$x_1 = \frac{hf - bg}{ab - h^2}, \quad y_1 = \frac{gh - af}{ab - h^2} \qquad (6)$$

$$(x_1, y_1) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

This is the centre of the conic (1) using (4) in (3) we get

$$ax'^2 + 2hx'y' + by'^2 + c_1 = 0 \qquad (5)$$

where

$$\begin{aligned} c_1 &= ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \\ &= ax_1^2 + hx_1y_1 + gx_1 + hx_1y_1 + by_1^2 + fy_1 + gx_1 + fy_1 + c \\ &= x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + gx_1 + fy_1 + c \\ c_1 &= gx_1 + fy_1 + c \\ &= \frac{g(hf - bg) + f(gh - af) + c(ab - h^2)}{ab - h^2} \\ &= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\triangle}{ab - h^2} \end{aligned}$$

Then equation (5) becomes $ax'^2 + 2hx'y' + by'^2 = -\dfrac{\triangle}{ab - h^2}$.
This equation can be written as

$$Ax'^2 + 2Hx'y' + By'^2 = 1 \qquad (6)$$

This is equation of central conic.

**Example 5.5** Reduce the equation

$$5x^2 + 6xy + 5y^2 - 10x - 6y - 3 = 0 \qquad (1)$$

to the standard form and name the conic.

**Solution:** We have $a = 5, b = 5, h = 3, g = -5, f = -3, c = -3$.

$$\begin{aligned} \triangle &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= -75 + 90 - 45 - 125 + 27 = -128 \neq 0 \end{aligned}$$

Equation (1) represents a conic. As $h^2 - ab = 9 - 25 = -16 < 0$ equation (1) represents ellipse. Centre of ellipse is $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right) = (1, 0)$
Shift the origin to the point $(h, k) = (1, 0)$

$$x = x' + h = x' + 1, \quad y = y' + k = y' \qquad (2)$$

Using (2) in (1)

$$\begin{aligned} 5(x' + 1)^2 + 6(x' + 1)y' + 5y'^2 - 10(x' + 1) - 6y' - 3 &= 0 \\ 5x'^2 + 6x'y' + 5y'^2 &= 8 \quad (3) \end{aligned}$$



Rotate the co-ordinate axes through an angle $\theta = \frac{\pi}{4}$ as $a = b$. Then the equations of rotations becomes

$$\begin{aligned} x' = x' \cos\theta - y'' \sin\theta &= x'' \cos\frac{\pi}{4} - y'' \sin\frac{\pi}{4} = \frac{x'' - y''}{\sqrt{2}} \\ y' = x'' \sin\theta + y'' \cos\theta &= x'' \sin\frac{\pi}{4} + y'' \cos\frac{\pi}{4} = \frac{x'' + y''}{\sqrt{2}} \qquad (4) \end{aligned}$$

Using (4) in (3)

$$5\frac{(x'' - y'')^2}{2} + 6\frac{(x'' - y'')(x'' + y'')}{2} + 5\frac{(x'' + y'')^2}{2} = 8$$

$$5(x''^2 - 2x''y'' + y''^2) + 6(x''^2 - y''^2) + 5(x''^2 + 2x''y'' + y''^2) = 16$$

$$16x''^2 + 4y''^2 = 16 \text{ i.e.} \frac{x''^2}{1^2} + \frac{y''^2}{2^2} = 1.$$

This is the equation of an ellipse.

**Example 5.6** Reduce the following equation to its standard form

$$x^2 + 2xy + y^2 - 6x - 2y + 4 = 0 \qquad (1)$$

**Solution:** Here $a = 1, b = 1, c = 4, f = -1, g = -3, h = 1$.
Since $h^2 - ab = 0$, the equation (conic) represents a parabola. Rotate the axes through an angle $\theta$ so that product term $xy$ will be eliminated from (1). Since, $a = b$, $\theta = \frac{\pi}{4}$. The equations of transformations will be $x = \dfrac{x' - y'}{\sqrt{2}}$, $y = \dfrac{x' + y'}{\sqrt{2}}$. Equation (1) will be transformed to

$$(x' - \sqrt{2})^2 = \sqrt{2}y' \qquad (2)$$

Shift the origin to the point $(\sqrt{2}, 0)$. Then equation (2) becomes

$$x''^2 = -\sqrt{2}y'.$$

**Example 5.7** Shift the origin to the centre of the conic and then remove the product term $xy$ $x^2+4xy+y^2-2x+2y-6=0$ (1)
**Solution:** Here $a=1, b=1, c=-6, h=2, g=-1, f=1$. Since $h^2-ab=4-1=3>0$. Therefore equation (1) represents hyperbola.

$$\text{The centre of hyperbola} = \left(\frac{hf-bg}{ab-h^2}, \frac{gh-ab}{ab-h^2}\right) = (-1,+1).$$

Shift the origin to the point $(-1,1)$. The equations of transformation are

$$x = x'-1, \quad y = y'+1 \qquad (2)$$

Using equations (2) in equation (1) we get

$$x'^2+4x'y'+y'^2+4=0 \qquad (3)$$

We rotate the co-ordinate axes through an angle $\theta$ so as to eliminate the product term. Since, $a=b$ we get $\theta=\frac{\pi}{4}$. Then the equations of rotations are

$$\begin{aligned} x' &= x''\cos\theta - y''\sin\theta = x''\cos\frac{\pi}{4} - y''\sin\frac{\pi}{4} \\ y' &= x''\sin\theta + y''\cos\theta = x''\sin\frac{\pi}{4} + y''\cos\frac{\pi}{4} \\ x' &= \frac{x''-y''}{\sqrt{2}}, \quad y = \frac{x''+y''}{\sqrt{2}} \qquad (4) \end{aligned}$$

Using above equations (4) in equations (3)

$$\left(\frac{x''-y''}{\sqrt{2}}\right)^2 + 4\left(\frac{x''-y''}{\sqrt{2}}\right)\left(\frac{x''+y''}{\sqrt{2}}\right) + \left(\frac{x''+y''}{\sqrt{2}}\right)^2 + 4 = 0$$

$$\text{Thus, } 6x''^2-2y''^2+8=0 \text{ i. e. } -\frac{x''^2}{(\frac{2}{\sqrt{3}})^2}+\frac{y''^2}{2^2}=1.$$

This is equation of the hyperbola.

## Exercise 2

1. Reduce the following equations to its standard form.



   (i) $5x^2+6xy+5y^2-4x+4y-4=0$.
   (ii) $5x^2+6xy+5y^2-10x-6y-3=0$.
   (iii) $x^2+2xy+y^2-2x-1=0$.
   (iv) $5x^2-6xy+5y^2+18x-14y+9=0$.
   (v) $x^2+4xy+y^2-2x+2y-6=0$.
   (vi) $7x^2-6xy+7y^2+30x+10y+35=0$.
   (vii) $x^2-2xy+y^2-6x-2y+4=0$.

2. Find the centres of the following conics.

   (i) $x^2-4xy-2y^2+10x+4y=0$.
   (ii) $x^2-5xy+y^2+8x-20y+15=0$.
   (iii) $5x^2+6xy+5y^2+22x-6y+21=0$.

3. Determine the nature of the following conics

   (i) $x^2-xy+2y^2-2x-6y+7=0$
   (ii) $x^2+y^2-8x-6y+5=0$.
   (iii) $3x^2-8xy-3y^2-10x-4y+2=0$
   (iv) $y^2+4x+4y+16=0$.

4. Remove the product term from the following equations.

   (i) $5x^2+3xy+y^2+x-y-2=0$.
   (ii) $4x^2+6xy+4y^2-2x+2y+3=0$.

5. Show that the equation $5x^2+6xy+5y^2-10x-6y-3=0$, represents the ellipse. Find its centre, lengths of axes, equations of axes and length of latus rectum.

6. Show that the equation $9x^2-6xy+y^2-14x-2y+12=0$ represents a parabola. Find its vertex and latus rectum.

7. Determine the nature of the conic $x^2+12xy-4y^2-6x+4y+9=0$.

8. Discuss the nature of the conic $x^2 - 4xy - 2y^2 + 10x + 4y = 0$.

9. Show that the equation $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$ represents a parabola. Also show that

   (i) vertex is $\left(\frac{3}{5}, \frac{1}{5}\right)$. (ii)Latus rectum is $\frac{4}{\sqrt{5}}$.

   (iii) Focus is $\left(\frac{4}{5}, \frac{3}{5}\right)$. (iv) Axis of parabola is $2x - y - 1 = 0$

   (v) The equation of directrix is $x + 2y = 0$.

## Answers

1. (i) $4x''^2 + y''^2 = 4$. (ii) $x''^2 + 4y''^2 = 1$ (iii) $x''^2 = \frac{-1}{\sqrt{2}} y''$.
   (iv) $x''^2 + 4y''^2 = 4$. (v) $3x''^2 - y''^2 = 1$. (vi) $\frac{x''^2}{5} + \frac{y''^2}{2} = 1$.
   (vii) $x''^2 = -\sqrt{2}y'$.

2. (i) $(-1, 2)$ (ii) $(-4, 0)$ (iii) $(-4, 3)$

3. (i) Ellipse (ii) Circle (iii) Hyperbola (iv) Parabola

5. (i) $(1, 0)$, semi-major axis $= 2$, semi-minor axis $= 1$. Equations of major axis $x + y + 1 = 0$, Equations of minor axis $x - y - 1 = 0$, length of latus rectum $= 1$.

6. Vertex $= (1, 1)$, latus rectum $= \frac{2}{\sqrt{10}}$. 7, 8. Hyperbola.

## Exercise 3

1. The origin is changed to the point $(h - 1)$. Determine the value of $h$ so that the new equation of the locus given by $2x^2 + 4x + 3y - 7 = 0$ will not contain first degree term in $x$.

2. The origin is changed to the point $(-2, k)$. Determine the value of $k$ so that the new equation of locus given by $2y^2 + 3x + 4y = 0$ will not contain first degree term in $y$.

3. Find the form of the equation $2x^2 + 3xy - 4y^2 + x + 3 = 0$ when origin is shifted to the point $(-2, 1)$.



4. Shift the origin to the point $(-1, 2)$ and transform the equation $x^2 + y^2 + 2x + 4y = 0$.

5. Shift the origin to a suitable point so that $x^2 + 4x - 8y + 12 = 0$ will be in the form $x^2 = 4by$. State the value of $b$.

6. Shift the origin to a suitable point so that the equation $x^2 - 6x - 4y + 1 = 0$ will be in the form $x^2 = 4ay$. State the value of $a$.

7. Under the translation of axes, the equation $2x^2 - 3y^2 + 4y + 5 = 0$ is transformed into $2x'^2 - 3y'^2 + 4x' - 8y' + 3 = 0$. Find the coordinates of new origin w.r.t. old origin.

8. The origin is shifted to the point $(h, 2)$. Find the value of $h$ so that the transformed equation of locus given by the equation $x^2 + 4x + 3y - 5 = 0$ will not contain a first degree term in $x$.

9. Transform the equation $3x^2 + 2xy + 3y^2 + 8x + 3y + 4 = 0$ by rotating the axes through an angle $\theta$ where $\theta = \sin^{-1}\left(\frac{3}{5}\right), 0 < \theta < \frac{\pi}{2}$, keeping the origin same.

10. Find the transformed form of the equation $x^2 + 4xy + y^2 = 0$ when the axes are rotated through an angle $\theta = \tan^{-1}(3)$ without changing the origin.

11. Transform the equation $11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ by shifting the origin to the point $(2, -1)$ inclined at an angle $\tan^{-1}\left(\frac{-4}{3}\right)$ to the original axes.

12. If by rotation of axes without changing the origin the equation $ax^2 + 2hxy + by^2 = 0$ becomes $a'x'^2 + 2h'x'y' + b'y'^2 = 0$, then show that $(a - b)^2 + 4h^2 = (a' - b')^2 + 4h'^2$.

13. Transform the equation $4x^2 + 2\sqrt{3}xy + 2y^2 - 1 = 0$ by rotating the axes through an angle of $30°$.

14. If under rotation of axes, without shifting the origin, the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is transformed to $a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c'$ then show that $g^2 + f^2 = g'^2 + f'^2$.

15. What does the equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ becomes under shifting the origin to the point $(2, 3)$ followed by rotation of the axes through $\frac{\pi}{4}$?

16. Transform the equation $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$ by shifting the origin to the point $(-4, 0)$ and then rotating the axes through an angle $45°$.

17. Find the angle $\theta$ through which the axes should be rotated to remove the $xy$ term in the following equations.
(i) $x^2 - 4xy + 4y^2 - 2y + 2 = 0$
(ii) $7x^2 + 12xy - 5y^2 + 4x + 3y - 5 = 0$
(ii) $4x^2 + 12xy + 9y^2 + 2x + 2y + 7 = 0$
(iv) $36x^2 + 28xy + 2gy^2 + 8x + 3y + 9 = 0.$
(v) $8x^2 - 12xy + 17y^2 + 4x + 6y - 2 = 0.$

18. Remove the product term $xy$ from the following equations:
(i) $3x^2 - 5xy + 3y^2 - 5 = 0$ (ii) $4x^2 + 2\sqrt{3}xy + 2y^2 - 7 = 0.$

## Answers

1. $h = -1$ 2.$k = -1$ 3. $2x^2 + 3xy - 4y^2 - 4x - 14y - 1 = 0.$

4. $x^2 - y^2 + 3 = 0.$ 5. $(-2, 1), b = 2.$ 6. $(3, -2), a = 1.$

7. $(1, 2)$. 8. $h = -2.$

9. $99x'^2 - 14x'y' + 51y'^2 + 205x' - 60y' + 100 = 0.$

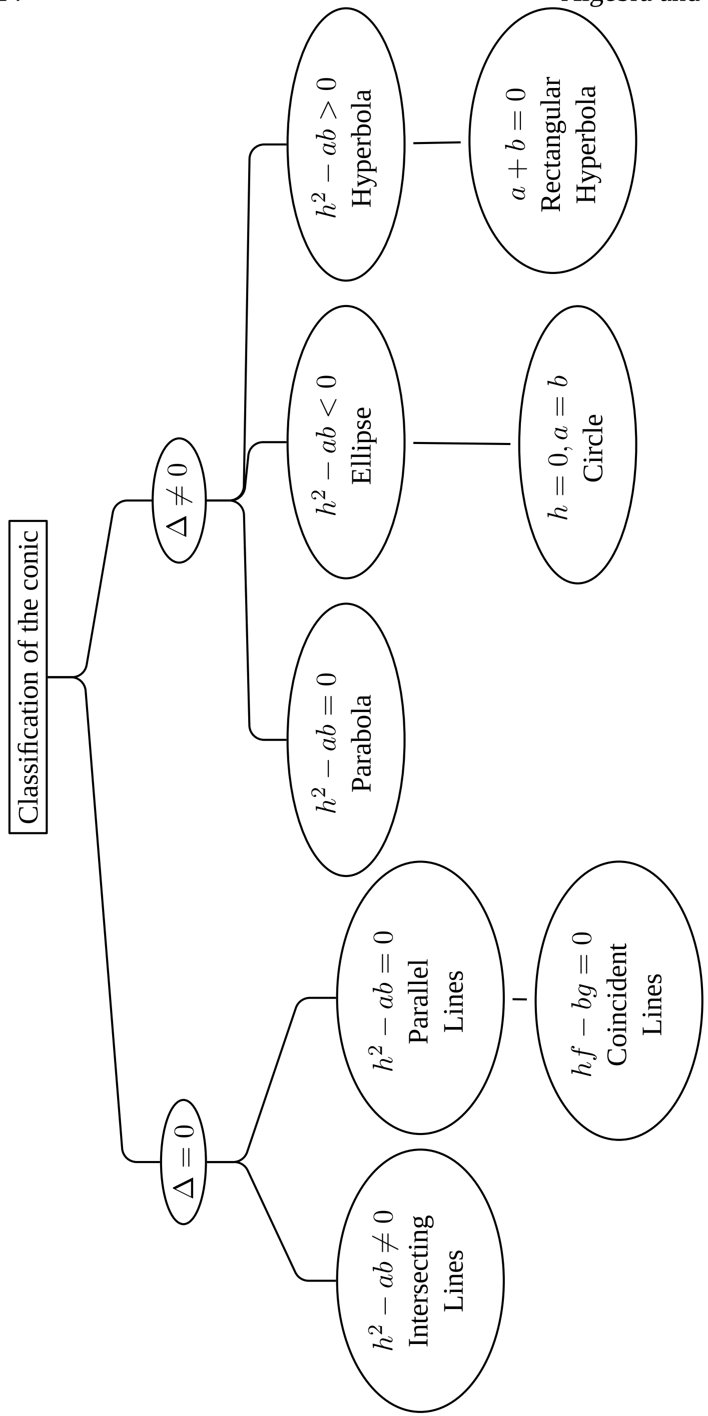
10. $11x'^2 - 16x'y' - y'^2 = 0.$ 11. $x''^2 - 4y''^2 + 1 = 0.$

13. $5x'^2 + y'^2 = 1.$ 15. $4x''^2 + 2y''^2 = 1.$ 16. $7x''^2 - 3y''^2 = 2.$

17. (i) $\frac{1}{2}\tan^{-1}\frac{4}{3}$. (ii) $\frac{\pi}{8}$ (iii) $\frac{1}{2}\tan^{-1}\left(\frac{-12}{5}\right)$.
(iv) $\frac{1}{2}\tan^{-1}(4)$. (v) $\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)$.

18. (i) $x'^2 + 11y'^2 = 10$ (ii) $5x'^2 + y'^2 = 7.$





# Chapter 6

# Planes in three dimensions

## 6.1 Introduction

The reader has already been introduced to the study of three dimensional geometry through vector methods. In this chapter, we study planes. We determine the equation of planes in different form. Note that in sections 6.2 to 6.8, we take the revision of known results.

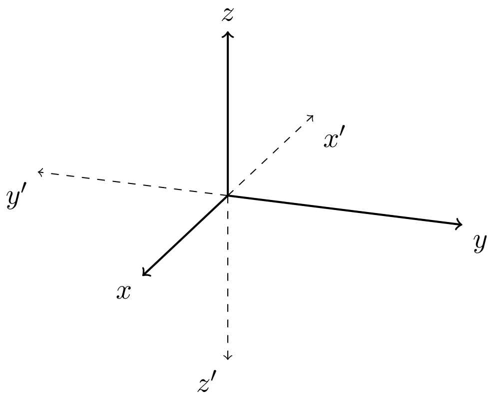



Fig. 6.1

## 6.2 Rectangular Cartesian Co-ordinates of a point in Space

In a plane we determine the position of a point by means of a pair rectangular axes. A point in the plane is identified with an ordered pair of real numbers called *the coordinates of the point*. We now extend this idea to the points in space. Let $X'OX$, $Y'OY$ and $Z'OZ$ be three mutually perpendicular axes intersecting in a point $O$. The point $O$ is called as the origin. The axes $X'OX$, $Y'OY$ and $Z'OZ$ are respectively called as the





$x$- axis, $y$-axis and $z$ -axis and will be referred as $the\ coordinate\ axes$. These three co-ordinate axes taken two by two, determines three mutually perpendicular planes. These three planes are called $the\ coordinate\ planes$ and are briefly written as the $xy$ - plane, $yz$ - plane, $zx$ - plane.

### 6.2.1 Orientation of Axes

The positive direction of the $z$-axis is the direction in which a right handed screw will move if the sense of rotation of the screw is from the positive direction of the $x$-axis to the positive direction of the $y$-axis.

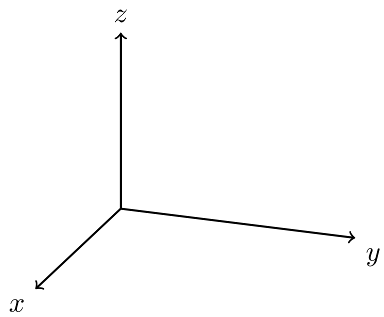



Fig . 6.2

Thus, $OX$, $OY$ and $OZ$ are the positive directions of the co-ordinate axes (see Fig 6.2). We then say that the co-ordinate system is oriented as a right handed system.
NOTE: We always use right handed rectangular coordinate system.

### 6.2.2 Co-ordinates of a Point

In order to define the coordinates of a point in space, we need the following definition. A line $L$ said to be perpendicular to a plane $\pi$, if it is perpendicular to every line contained in $\pi$. It can be shown that the line $L$ is perpendicular to $\pi$ if and only if $L$ is perpendicular to two distinct intersecting lines in $\pi$. Let P be any point in the space (see Fig 6.3). Draw $seg\ PN$ perpendicular on the $xy$-plane. Draw $seg\ NA$ and $seg\ NB$ perpendicular to the $x$ and $y$ axes respectively. Draw also, $seg\ PC$ perpendicular to the $z$-axis.

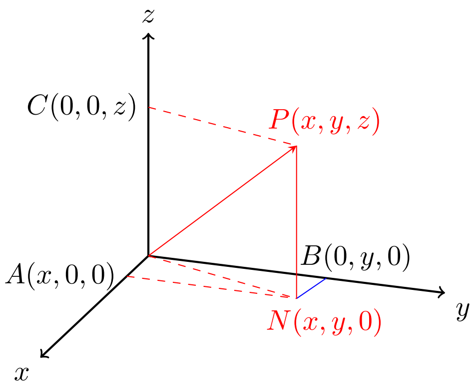



Fig. 6.3

Thus, corresponding to every point in the space we can find three points $A, B, C$ on the coordinate axes. We now call the directed lengths $OA$, $OB$ and $OC$ the $x, y$ and $z$ coordinates of the point $P$. We denote it as $P(x, y, z)$.

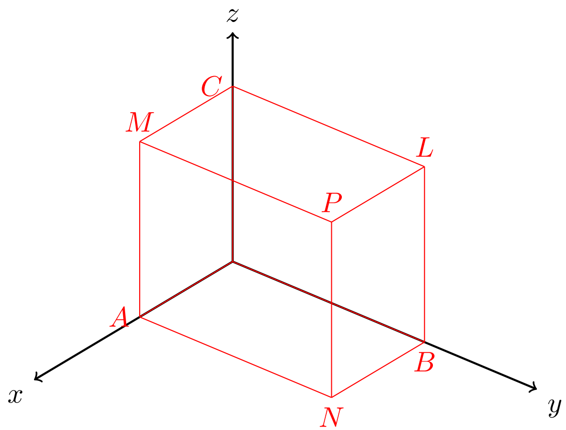



Fig. 6.4



**Remark 6.1** If we draw planes parallel to the coordinate axes passing through $P$, we get a parallelepiped $OANB - CMPL$ $(see\ Fig.6.4)$. $PM, PL, PN$ are perpendiculars on $yz$, $zx$ and $xy$ panes respectively. These planes intersect $x, y, z$ axes in $A, B, C$ respectively. By, $x, y, z$ co-ordinates of $P$ we mean real numbers $x, y, z$ such that $x = OA$, $y = OB$, $z = OC$. Now, $PN \perp xy$-plane, therefore $PN$ is perpendicular to every line in the $xy$-plane. Hence, $PN \perp OX$, also $NA \perp OX$. So $OX$ is perpendicular to two intersecting lines $PN$ and $NA$. Thus, $OX$ is perpendicular to the plane formed by $PN$ and $NA$. Thus, $OX$ is perpendicular to every line in the plane. Hence, $OX \perp PA$, $(see\ Fig.6.4)$. Similarly, $OY \perp PB$ and $OZ \perp PC$. Hence, $x, y$ and $z$ are the projections of $OP$ on $OX$, $OY$ and $OZ$ respectively. If $OP$ make angles $\alpha$, $\beta$, $\gamma$ with axes $OX, OY$ and $OZ$ respectively; and $OP = r$, then

$$\begin{aligned} x &= OA = \text{projection of } OP \text{ on the } x\text{-axis } = r\cos\alpha \\ y &= OB = \text{projection of } OP \text{on the} y\text{-axis } = r\cos\beta \\ z &= OC = \text{projection of } OP \text{ on the} z\text{-axis} = r\cos\gamma. \end{aligned}$$

### 6.2.3 Direction Cosines

In plane geometry the direction of a line is determined by its inclination $\theta$ $(0 \le \theta \le \pi)$; i.e., angle $\theta$ made by a line with positive direction of the $x - axis$. To determine the direction of a line in space, we must know the angles made by the line with $x, y$ and $z$ axes. Now on a line $AB$ there are two possible directions $viz.$ $\overline{AB}$ and $\overline{BA}$. Hence to be definite we consider direction of vectors.

**Definition 6.1 (Direction angles)** Let $\overline{OP}$ represent a vector $\bar{r}$ in space. Let $\alpha, \beta, \gamma$ be the angles made by $\bar{OP}$ with positive directions of $x, y, z$ axes respectively, with $0 \le \alpha, \beta, \gamma \le \pi$. Then $\alpha, \beta, \gamma$ are called the direction angles of the vector $\bar{r}$.

**Definition 6.2 (Direction cosines)** If $\alpha, \beta, \gamma$ are direction angles of a vector $\bar{r}$, then $\cos\alpha, \cos\beta, \cos\gamma$ are called $the\ direction\ cosines\ of\ \bar{r}$.
We shall write in short $d.c.s.$ for direction cosines. Further it is customary to write the $d.c.s.$ as $l, m, n$ (where, $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$).

**Remark 6.2** If $l, m, n$ are $d.c.s.$ of $\bar{r}$, then $-l, -m, -n$ are $d.c.s.$ of $-\bar{r}$, direction angles of $-\bar{r}$ are $\pi - \alpha, \pi - \beta, \pi - \gamma$.

**Definition 6.3 (Direction cosines of a line)** Let $L$ be a straight line in the space and $A, B$ be points on it. Suppose $P$ is a point on the straight line parallel to the line $L$, passing through the origin, such that $\overline{AB}$ and $\overline{OP}$ have the same direction. The direction angles of the line $L$ are defined to be the direction angles of the vector $\overline{OP}$. If $\alpha, \beta, \gamma$ are direction angles of a line $L$, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines $(d.c.s.)$ of $L$.

**Remark 6.3** If $\alpha, \beta, \gamma$ are direction angles of a line $L$, then $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$ are also direction angles of the line $L$. Thus, if $\cos\alpha, \cos\beta, \cos\gamma$ are $d.c.s.$ of the line $L$, then $-\cos\alpha, -\cos\beta, -\cos\gamma$ are also $d.c.s.$ of $L$. Any one of these can be used as $d.c.s.$ of the line $L$.

**Remark 6.4** *Two or more lines are parallel if and only if they have the same sets of* $d.c.s.$.

Direction cosines of the coordinate axes Direction angles of the $x$-axis are $0, \pi/2, \pi/2$. Therefore $d.c.s.$ of the $x$-axis are $\cos 0, \cos\pi/2, \cos\pi/2$ i.e. $1, 0, 0$. Similarly, the $d.c.s.$ of $y$ and $z$ axes are $0, 1, 0$ and $0, 0, 1$ resp.

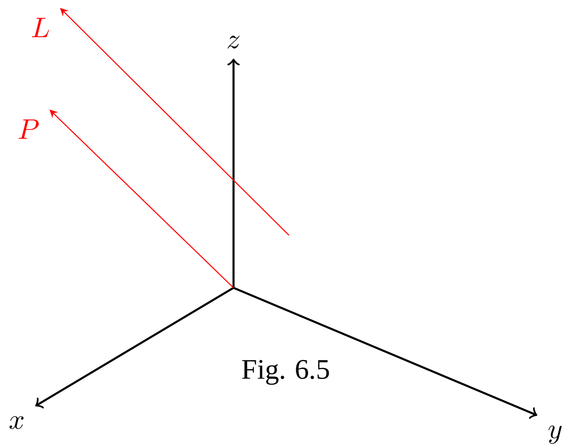


Fig. 6.5



**Theorem 6.1** *If $l, m, n$ are $d.c.s.$ of a line the $l^2 + m^2 + n^2 = 1$.*

**Proof.** *Let $L$ be a line with $d.c.s.$ $l, m, n$. Let a line $OP$ be drawn through the origin $0$ and parallel to the line $L$, where $P(x, y, z)$. The line $L$ and the line $OP$ make the same angles with the coordinate axes. Hence the $d.c.s.$ of the line $OP$ will also be $l, m, n$. Let $\alpha, \beta, \gamma$ be the direction angles of the line $OP$ (see Fig. 6.5). Therefore $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$. Let $OP = r$. By the* ***Remark 6.1****, we have $x = r\cos\alpha, y = r\cos\beta, z = r\cos\gamma$. By the distance formula, we have $x^2 + y^2 + z^2 = r^2$. Hence, $r^2\cos^2\alpha + r^2\cos^2\beta + r^2\cos^2\gamma = r^2$. Thus,*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \text{ i.e. } l^2 + m^2 + n^2 = 1. \quad \blacksquare$$

**Definition 6.4 (Direction Ratios)** If $l, m, n$ are $d.c.s.$ of a line, then any three numbers $a, b, c$ which are proportional; i.e., $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$ are called *the direction ratios* $(d.r.s.)$ of the line.

Note that $a, b, c$ are $d.r.s.$ of a line $L$, then so are $ka, kb, kc$ for nonzero real number $k$.

**Remark 6.5** In the proof of the Theorem 6.1, we have $l = \frac{x}{r}, m = \frac{y}{r}$, $n = \frac{z}{r}$. So, for a point $P(x, y, z)$, the $d.c.s.$ of the line $OP$ ($O$ is the origin) are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$. This means that $x, y, z$ are proportional to the $d.c.s.$ of the line $OP$. It follows that the coordinates $(x, y, z)$ of P are $d.r.s.$ of the line $OP$.

**Remark 6.6 (Direction ratios of a line joining two points.)** *Let $PQ$ be a line passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Shift the origin to the point $P$. The coordinates of $Q$ will then be $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. But $P$ is the new origin and so the line $PQ$ passes through the new origin. Hence by the* ***Remark 6.5*** *$d.r.s.$ of $PQ$ are precisely the new coordinates of $Q$ $viz.$ $x_2 - x_1, y_2 - y_1, z_2 - z_1$.*

**Remark 6.7 (Relation between direction ratios and direction cosines)** Let $l, m, n$ be the $d.c.s$ and $a, b, c$ are $d.r.s.$ of a line $L$. By definition of $d.r.s.$ we have, $l = at$, $m = bt$ and $n = ct$ for some t$\in\mathbb{R}$. Since, $l^2 + m^2 + n^2 = 1$, we have, $t^2(a^2 + b^2 + c^2) = 1$. Hence,

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Note that the same plus or minus sign has been taken through out.

### 6.2.4 Angle between two lines

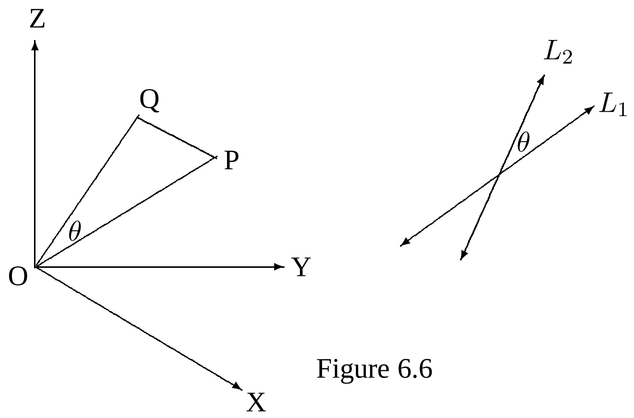



Figure 6.6

Let $L_1$ and $L_2$ be two lines in the space which make an angle $\theta$ with each other. Through origin $O$ draw lines $OP$ and $OQ$ parallel to line $L_1$ and $L_2$ respectively, where $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. By the distance formula we have,

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \tag{1}$$

Let $l_1, m_1, n_1$ be $d.c.s.$ of the line $OP$ and $l_2, m_2, n_2$ be $d.c.s.$ of the line $OQ$. If $OP = r_1$ and $OQ = r_2$, then by ***Remark 6.1*** we have $x_1 = l_1 r_1$, $y_1 = m_1 r_1$, $z_1 = n_1 r_1$; $x_2 = l_2 r_2$, $y_2 = m_2 r_2$, $z_2 = n_2 r_2$. Substitute these values in (1), we get

$$\begin{aligned} PQ^2 &= (l_1 r_1 - l_2 r_2)^2 + (m_1 r_1 - m_2 r_2)^2 + (n_1 r_1 + n_2 r_2)^2 \\ \therefore \quad PQ^2 &= (l_1^2 + m_1^2 + n_1^2) r_1^2 + (l_2^2 + m_2^2 + n_2^2) r_2^2 \\ &\quad -2(l_1 l_2 + m_1 m_2 + n_1 n_2) r_1 r_2. \end{aligned}$$

As, $l_1^2 + m_1^2 + n_1^2 = 1$ and $l_2^2 + m_2^2 + n_2^2 = 1$, we have,

$$PQ^2 = r_1^2 + r_2^2 - 2(l_1 l_2 + m_1 m_2 + n_1 n_2) r_1 r_2$$



By the cosine rule of trigonometry, $PQ^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta$.

$$\therefore \quad \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

This gives an expression for the angle between two lines.

**Remark 6.8 (Condition for two lines to be perpendicular)** If the two lines are perpendicular, then the angle between them, say $\theta$ is $90°$, so that $\cos\theta = 0$. Hence, we get the condition, $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

If the direction ratios of the lines are $a_1, b_1, c_1$ and $a_2, b_2, c_2$, then from ***Remark 6.8*** the condition for perpendicularity becomes,

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$$

**Remark 6.9 (Condition for two lines to be parallel)** *If two lines are parallel make the same angle with each of the coordinate axes. Hence, the two lines will be parallel, if and only if $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$. If the $d.r.s.$ of the lines are $a_1, b_1, c_1$ and $a_2, b_2, c_2$, then the lines will be parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.*

## 6.3 General Equation of First Degree

An equation of the first degree in $x, y, z$ is of the form $ax + by + cz = 0$, where $a, b, c, d$ are given real numbers and $a, b, c$ are not all zero, simultaneously.

A surface is called $a\ plane$, if given any two points on the surface, then a straight line joining them also lies completely on the surface; i.e. if $A$ and $B$ are any points on the surface and $P$ is any point on the line $AB$, then $P$ also lies on the surface.

**Theorem 6.2** Every equation of first degree in $x, y, z$ represents a plane.

**Proof.** Consider the first degree equation in $x, y, z$,

$$ax + by + cz + d = 0 \tag{2}$$

where the coefficients $a, b, c$ are not all zero.

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points on the locus given by the equation (2). Then,

$$ax_1 + by_1 + cz_1 + d = 0 \quad (3)$$
$$ax_2 + by_2 + cz_2 + d = 0 \quad (4)$$

Let $P$ be any point on the line $AB$. we show that $P$ lie on the locus (2). Suppose $P$ divides $AB$ in the ratio $m : n$. By the section formula, we get coordinates of $P$ as, $\left(\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}, \dfrac{mz_2+nz_1}{m+n}\right)$. Now,

$$a\left(\frac{mx_2+nx_1}{m+n}\right) + b\left(\frac{my_2+ny_1}{m+n}\right) + c\left(\frac{mz_2+nz_1}{m+n}\right)$$
$$= \frac{m(ax_1+by_1+cz_1+d)+n(ax_2+by_2+cz_2+d)}{m+n} = 0$$

using equation (3) and (4)). It follows that the coordinates of $P$ also satisfy (2). As $P$ is any point on the line $AB$. It follows that the line$AB$ lies on the locus (2). It can be proved that the set

$$\{(x,y,z) \in \mathbb{R}^3 | ax+by+cz+d=0\}$$

contains at least three non-collinear points; i.e. the locus (2), is not a straight line; also the locus (2), is not $\mathbb{R}^3$. Therefore the general equation of the first degree in $x, y, z$ represents a plane. ∎

Note, that the property, if $A$ and $B$ are two points in the set then all points on the straight line through $A$ and $B$ are in the set; is true for a straight line and three dimensional space also.

## 6.4 Normal form of the equation of a plane

To find the equation of a plane in terms of $p$, the length of the perpendicular to it from the origin and $l, m, n$ the $d.c.s.$ of this perpendicular. Let $\pi$ be the plane (see Fig. 6.7) whose equation is required. Draw $ON$ perpendicular to the plane $\pi$ from the origin $O$. Let $ON = p$. The $d.c.s.$ of $ON$ are $l, m, n$. Observe that if $(x', y', z')$ are coordinates of $N$, then $x', y', z'$ are $d.r.s.$ of $ON$ $(see$ ***Remark 6.5***$)$



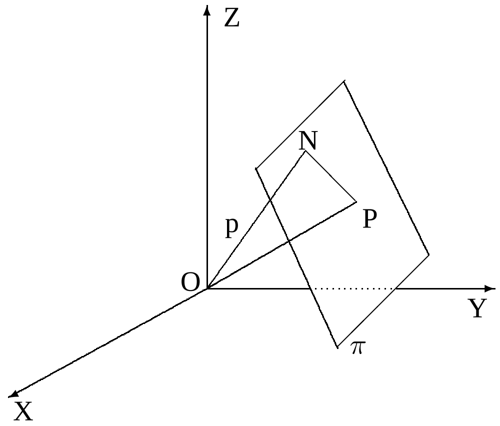



Figure 6.7

$$\therefore \quad \frac{x'}{l} = \frac{y'}{m} = \frac{z'}{n} = p,$$

So that $x' = lp, \quad y' = mp, z' = np$. Hence the coordinates of $N$ are $(lp, mp, np)$. Let $P(x,y,z)$ be any point on the plane $\pi$. The of $d.r.s.$ $NP$ are $x - lp, y - mp, z - np$ $(see$ ***Remark 6.6***$)$. But $ON$ is perpendicular to $NP$, because $NP$ lies in the plane $\pi$ and $ON$ is perpendicular to $\pi$. Hence $(see$ ***6.8***$)$,

$$l(x-lp) + m(y-mp) + n(z-np) = 0$$
$$\therefore \quad lx + my + nz - p(l^2+m^2+n^2) = 0$$
$$\therefore \quad lx + my + nz = p, \quad since \quad l^2+m^2+n^2 = 1.$$

This equation is called *the normal form of the equation of a plane*.

Note: From this result, we observe that the equation of any plane is a linear equation in $x, y, z$. This is the converse of the result that, every equation of the first degree in $x, y, z$ represents a plane.

## 6.5 Transform to the normal form

To transform the equation $ax + by + cz + d = 0$ of a plane $\pi$ to the normal form $lx + my + nz = p$. Observe that $l, m, n$ are $d.c.s.$ of a normal to the plane $\pi$ and $p$ is the length of the perpendicular from the origin on it. As these two equations represent the same plane $\pi$, we have,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{-p}{d} = \frac{\pm\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}};$$

i.e. $$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{-p}{d} = \frac{\pm 1}{\sqrt{a^2+b^2+c^2}}.$$ Thus,

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

The positive or negative sign in $\pm\sqrt{a^2+b^2+c^2}$ is chosen such that $p$ is always positive. Thus, the normal form of the plane $ax+by+cz+d=0$ is,

$$\frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}} = \pm\left(\frac{-d}{\sqrt{a^2+b^2+c^2}}\right)$$

**Remark 6.10**

1. We note that the coefficients $a, b$ and $c$ of $x, y$ and $z$ in $ax+by+cz+d=0$ are $d.r.s.$ of a normal to the plane.
2. It also follows that the length of the perpendicular from the origin on the plane is, $\frac{-d}{\pm\sqrt{a^2+b^2+c^2}}$.

## 6.6 Angle between two planes

Angle between two planes is equal to the angle between their normals from any point. Let

$$ax_1+by_1+cz_1+d_1 = 0 \quad (5)$$

and $$ax_2+by_2+cz_2+d_2 = 0 \quad (6)$$

be the equations of the two planes which intersect each other. Observe that $d.r.s.$ of a normal to (5)are $a_1, b_1, c_1$ and of (6) are $a_2, b_2, c_2$. If $\theta$ is the angle between the planes and hence between that normals then, we have

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$



The two plane are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$ and parallel if, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

## 6.7 Determination of a plane under given conditions

Intercept form of the equation of a plane. To find the equation of a plane which makes intercepts $a, b$ and $c$ respectively on the coordinate axes $x, y$ and $z$. Let,

$$Ax+By+Cz+D=0 \quad (7)$$

be the equation of the plane which makes intercepts $a, b, c$ on the coordinate axes. Then the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ lie on the plane. The coordinates of each point satisfy. Thus $A.a+B.b+C.c+D=0$ gives $A=-D/a$. Similarly, we get $B=-D/b$, $C=-D/c$. The equation (7) can be written as,

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0$$

$$\therefore \quad D\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right) = D$$

Observe that $D \neq 0$ for otherwise the plane would pass through the origin and there would be no intercepts on the axes. Hence $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is the required equation of plane.

## 6.8 Plane passing through a given point

To find the equation of a plane passing through the point $A(x_1,y_1,z_1)$ and $d.r.s.$ of whose normal are $a, b, c$. Let the equation of the plane be

$$ax+by+cz+d=0 \quad (8)$$

Coordinates of $A$ satisfy the equation (8), so we get $ax_1+by_1+cz_1+d=0$ so

$$d = -ax_1 - by_1 - cz_1.$$

Substitute the value of $d$ in (8), we have

$$\begin{aligned} ax+by+cz+(-ax_1-by_1-cz_1)&=0\\ \therefore\quad a(x-x_1)+b(y-y_1)+c(z-z_1)&=0\end{aligned}$$

This is the equation of the plane passing through the point $(x_1,y_1,z_1)$ and $d.r.s.$ of whose normal are $a,b,c$.

## 6.9 Plane passing through three points.

To find the equation of the plane passing through three non collinear points $(x_1,y_1,z_1)$, $(x_2,y_2,z_2)$, $(x_3,y_3,z_3)$. Let the equation of the plane be,

$$ax+by+cz+d=0 \tag{9}$$

As the given three point lie on the plane(9), their coordinates satisfy (9). Hence we have,

$$ax_1+by_1+cz_1+d=0 \tag{10}$$

$$ax_1+by_1+cz_1+d=0 \tag{11}$$

$$ax_1+by_1+cz_1+d=0 \tag{12}$$

All these equations are linear equations in $a,b,c,d$. Eliminating $a,b,c,d$ from these equation will give the required equation of the plane. Hence, we have the required equation as,

$$\begin{vmatrix} x & y & z & 1\\ x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\end{vmatrix}=0.$$

Using the elementary row transformations, $R_1-R_2, R_3-R_2, R_4-R_2$, we get

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 & 0\\ x_1 & y_1 & z_1 & 1\\ x_2-x_1 & y_2-y_1 & z_2-z_1 & 0\\ x_3-x_1 & y_3-y_1 & z_3-z_1 & 0\end{vmatrix}=\begin{vmatrix} x-x_1 & y-y_1 & z-z_1\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix}=0.$$



Equivalently, we can also use (8). Let the required equation of the plane be

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0.$$

Since $(x_2,y_2,z_2)$, $(x_3,y_3,z_3)$ lie on this plane, they satisfy the above equation. Thus,

$$\begin{aligned} a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1) &= 0,\\ a(x_3-x_1)+b(y_3-y_1)+c(z_3-z_1) &= 0. \end{aligned}\tag{13}$$

Solving (13) using Cramer's rule, we get

$$\frac{a}{\begin{vmatrix} y_2-y_1 & z_2-z_1\\ y_3-y_1 & z_3-z_1)\end{vmatrix}}=\frac{b}{\begin{vmatrix} z_2-z_1 & x_2-x_1\\ z_3-z_1 & x_3-x_1)\end{vmatrix}}=\frac{c}{\begin{vmatrix} x_2-x_1 & y_2-y_1\\ x_3-x_1 & y_3-y_1)\end{vmatrix}}$$

Substituting these values of $a,b$ and $c$, we get the equation of plane as

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix}=0.$$

## 6.10 Systems of Planes

The following are the equations of systems of planes containing one or two parameters.

1. We know that two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are parallel if and only if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$. Hence a plane parallel to the plane $ax+by+cz+d=0$ is given by $ax+by+cz+k=0$, $k\in\mathbb{R}$. For different values of $k$, we get the set of planes, each planes is parallel to given plane $ax+by+cz+d=0$. Here $k$ is called as the parameter. Thus $\{ax+by+cz+k=0|k\in\mathbb{R}\}$ represents the system of planes parallel to the given plane $ax+by+cz+d=0$.

2. Let $L$ be a line with $d.r.s\ a, b, c$. Then the equation of a plane perpendicular to the line $L$ is of the form $ax + by + cz + d = 0$, $d \in \mathbb{R}$. For different values of $d$, we get set of planes, each plane is perpendicular to the given line $L$. Thus

$$\{ax + by + cz + d = 0 | d \in \mathbb{R}\}$$

represents the system of planes perpendicular to the given line $L$ with d.r.s. $a, b, c$.

3. We know that the equation of a plane passing through the point $(x_1, y_1, z_1)$ is of the form $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where $a, b, c$ are real numbers not all zero simultaneously. Suppose $c \neq 0$. Then the equation reduces to
$A(x - x_1) + B(y - y_1) + (z - z_1) = 0$, where $A = \frac{a}{c}$, $B = \frac{b}{c}$.
For different values of $A$ and $B$, we get a set of planes, each plane of the set passes through the given point $(x_1, y_1, z_1)$. Here $A$ and $B$ are two parameters. Hence the equation $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ represents the system of planes passing through the point $(x_1, y_1, z_1)$, where the required two parameters are the ratios of the coefficients $a, b, c$.

4. Consider two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ which intersect each other in a line. Now consider the locus given by the following equation

$$(a_1x+b_1y+c_1z+d_1)+k(a_2x+b_2y+c_2z+d_2) = 0, k \in \mathbb{R}. \quad (14)$$

Rewrite this equation as
$(a_1 + ka_2)x + (b_1 + kb_2)y + (c_1 + kc_2)z + (d_1 + d_2) = 0.$
This is a linear equation in $x, y$ and $z$. Hence, it represents a plane. Thus equation (14) represents a plane. For different values of $k$, we get a set of planes. It is easy to see the plane given by (14) passes through the line of intersection of the planes $a_1x+b_1y+c_1z+d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. Hence the equation

$$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$$

represents the system of planes through the line of intersection of the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, $k$ being the parameter.



### 6.10.1 Two sides of a plane

It is clear that each plane $\pi$ divides the space into two parts, called the two sides or two half regions of the space determined by $\pi$. Two points $A$ and $B$, not on $\pi$, lie on different sides of $\pi$ if and only if the $segment\ AB$ intersects $\pi$ in a unique point, otherwise they lie on the same side of $\pi$.

**Theorem 6.3** *Two points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ lie on the same or different sides of the plane $ax + by + cz + d = 0$, if the expressions $ax_1 + by_1 + cz_1 + d$ and $ax_2 + b_2y + c_2z + d$ are of the same or different signs.*
**Proof.** Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points which are not on the plane

$$ax + by + cz + d = 0. \quad (15)$$

Let the line $AB$ meet the plane $(15)$ in the point $P$. Suppose $P$ divides $AB$ in the ratio $\lambda : 1$. If $\lambda$ is positive, then $P$ divides $AB$ internally; and $P$ divides $AB$ externally, if $\lambda$ is negative. By section formula, we get

$$P(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}).$$

$P$ lies on the plane $(15)$. Coordinates of $P$ satisfy the equation $(15)$. Therefore $a(\frac{\lambda x_2+x_1}{\lambda+1}) + b(\frac{\lambda y_2+y_1}{\lambda+1}) + c(\frac{\lambda z_2+z_1}{\lambda+1}) + d = 0$; i.e.,

$$(ax_1 + by_1 + cz_1 + d) + \lambda(ax_2 + by_2 + cz_2 + d) = 0.$$

From this equation, we get

$$\lambda = -\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}. \quad (16)$$

This shows that $\lambda$ is negative or positive according as $ax_1 + by_1 + cz_1 + d\ and\ ax_2+by_2+cz_2+d$ are of the same or different signs. Suppose $ax_1+by_1+cz_1+d\ and\ ax_2+by_2+cz_2+d$ are of the same signs. Therefore the

$\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$ is positive. Hence by $(16)$, $\lambda$ is negative. Therefore in this case $P$ divides $AB$ externally. Hence $A$ and $B$ lie on the same side of the plane $(15)$.

If $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the different signs, then $A$ and $B$ lie on the different sides of the plane $(15)$.

**Example 6.1** Show that the origin and the point $(2, -4, 2)$ lie on the different sides of the plane $x + 3y - 5z + 7 = 0$.

**Solution.** Let $\alpha$ denote the expression $x + 3y - 5z + 7$. The value of the expression $\alpha$ at the origin is $0 + 3(0) - 5(0) + 7 = 7 > 0$. The value of the expression $\alpha$ at the point $(2, -4, 3)$ is

$2 + 3(-4) - 5(3) + 7 = -18 < 0$. The values of the expressions have different signs. Hence the origin and the point $(2, -4, 3)$ lie on different sides of the plane $x + 37 - 5z + 7 = 0$.

## 6.11 Length of the perpendicular from a point to a plane.

Consider the equation of the plane in the normal form viz.

$$lx + my + nz = p \tag{17}$$

where $p$ denotes the length of the perpendicular from the origin to the plane; and $l, m, n$ are $d.c.s.$ of the normal to the plane.

The equation of a plane parallel to the plane (17) and passing through the point $P(x_1, y_1, z_1)$ is given by $l(x - x_1) + m(y - y_1) + n(z - z_1) = 0$. i.e.,

$$lx + my + nz = p_1, \tag{18}$$

where $p_1 = lx_1 + my_1 + nz_1$.

Let $OKK'$ be the perpendicular from the origin $O$ to the two parallel planes meeting them in $K$ and $K'$. If $p_1 > 0$ then $K$ and $K'$ are on the same side of the plane $lx + my + nz = 0$ so that $OK = p$ and $OK' = p_1$. Draw $PL$ perpendicular from $P$ to the plane given by (17) (see figure 6.8). We have $PL = K'K = OK' - OK = p_1 - p$. $P$ lies on the plane given by (18). As $lx_1 + my_1 + nz_1 = p_1$, we get $PL = lx_1 + my_1 + nz_1 - p$. Thus the



required length of the perpendicular from $P$ to the plane $lx + my + nz = p$ is $|PL| = |lx_1 + my_1 + nz_1 - p|$.

If $p_1 < 0$ then $K$ and $K'$ are on the opposite sides of the plane $lx + my + nz = 0$ so that $OK = p$ and $OK' = -p_1$. Then the distance between the two planes is $(-p_1) + p = p - (lx_1 + my_1 + nz_1)$. Thus, we get the distance between the planes as $|lx_1 + my_1 + nz_1 - p|$.

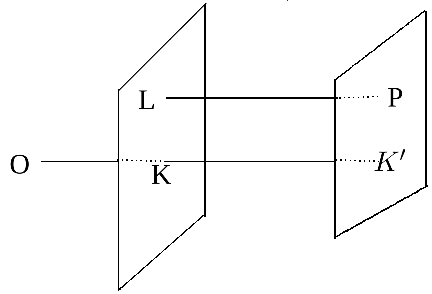



Figure 6.8

**Remark 6.11** The length of the perpendicular drawn from the point $(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

The normal form of the plane $ax + by + cz + d = 0$ is

$$\frac{ax + by + cz}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{-d}{\sqrt{a^2 + b^2 + c^2}},$$

the plus or minus sign being taken in the denominator according as d is negative or positive hence by $(5.10.1)$ the length of the perpendicular from $P$ to the plane $|\dfrac{ax_1 + by_1 + cz1 + d}{\pm\sqrt{a^2 + b^2 + c^2}}| = \dfrac{|ax_1 + by_1 + cz1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

**Example 6.2** Find the distance of the point $(1, 1, 4)$ from the plane $3x - 6y + 2z + 11 = 0$.

**Solution.** By ***Remark 6.11***, the distance of the point $(1, 1, 4)$ from the plane $3x - 6y + 2x + 11 = 0$ is $|\dfrac{3(1) - 6(1) + 2(4) + 11}{\sqrt{(3)^2 + (-6)^2 + (2)^2}}| = \dfrac{16}{7}$.

### 6.11.1 Bisectors of angles between two planes

*To find the equations of the planes bisecting the angles between the planes* $a_1x + b_1y + c_1z + d_1 = 0$ *and* $a_2x + b_2y + c_2z + d_2 = 0.$

Let $P(x, y, z)$ be a point on the plane bisecting the angle between the two given planes. Then the perpendicular distances from $P$ to the two given planes should be equal. Hence by ***Remark 6.11***,

$$|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}| = |\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}|.$$

Thus the equations of the bisecting planes are

$$\frac{ax_1 + by_1 + cz1 + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{ax_2 + by_2 + cz2 + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Of these two bisecting planes, one bisects the acute angle and other obtuse angle between the two given planes.

**Example 6.3** Find the equations of the planes bisecting the angles between the planes $x + 2y + 2z = 9$ and $4x - 3y + 12z + 13 = 0$. Also specify the one which bisects the acute angle.

**Solution.** The equations of the two bisecting planes are

$$\frac{x + 2y + 2z - 9}{\sqrt{1^2 + 2^2 + 2^2}} = \pm\frac{4x - 3y + 12z + 13}{\sqrt{4^2 + -3^2 + 12^2}};$$

i.e., $\frac{x+2y+2z-9}{3} = \frac{4x-3y+12z+13}{13}$ and $\frac{x+2y+2z-9}{3} = -\frac{4x-3y+12z+13}{13}$;i.e $x + 35y - 10z = 156$ and $25x + 17y + 62z = 78$.

To find which plane bisects the acute angle between the given planes, for this find the angle between $x + 2y + 2z = 9$ and one of these bisecting planes, say $x + 35y - 10z = 156$. Let $\theta$ be the angle between the planes $x + 2y + 2z = 9$ and $x + 35y - 10z = 156$.

$\therefore \quad \cos\theta = \frac{1(1)+2(35)+2(-10)}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+35^2+(-10)^2}} = \frac{17}{\sqrt{1326}}$. From $\cos\theta$, we find $\tan\theta = \frac{\sqrt{1037}}{17} > 1$. Therefore $\theta > 45°$. Hence the plane $x + 35y - 10z = 156$ bisects the obtuse angle between the given planes. This implies that the other plane, $25x + 17y + 62z = 78$ bisects the acute angle between the given planes.



## 6.12 Joint equation of two planes

Consider the equation of two planes

$$a_1x + b_1y + c_1z + d_1 = 0 \tag{19}$$
$$a_2x + b_2y + c_2z + d_2 = 0. \tag{20}$$

Then the joint equation of these two planes is given by

$$(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0 \tag{21}$$

Note that if $(x_1, y_1, z_1)$ lies on either the plane (19) or the plane (20), then either $a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$ or $a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$. Therefore $(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) = 0$. Hence the point $(x_1, y_1, z_1)$ lies on the locus given by (21). Conversely if the point $(x_1, y_1, z_1)$ lies on the locus given by (21), then $(x_1, y_1, z_1)$ lies on either the plane (19) or the plane (20). Thus we say that

$$(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) = 0$$

is the joint equation of the given two planes.

**Theorem 6.4** *The necessary and sufficient condition that the homogeneous second degree equation* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ *represents two planes is* $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

**Proof.** We suppose that the given equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \tag{22}$$

represent two planes. Let the equation of two separate planes be $lx + my + nz = 0$ and $l'x + m'y + n'z = 0$.

As the equation (22) is a homogenous equation, there can not appear constant terms in the separate equations of the plane. We have $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = (lx + my + nz)(l'x + m'y + n'z)$. Comparing coefficients,we get

$a = \lambda ll', b = \lambda mm', c = \lambda nn', 2f = \lambda nm' + mn', 2g = vln' + nl'$
$2h = vlm' + ml'$

$$\therefore \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = \lambda^3 \begin{vmatrix} ll' + ll' & lm' + ml' & ln' + nl' \\ lm' + ml' & mm' + mm' & nm' + mn' \\ ln' + nl' & nm' + mn' & nn' + nn' \end{vmatrix}$$

$= \lambda^3 \begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & m' & n' \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix} = 0$. Therefore, $8 \times \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Thus, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Conversely, suppose that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
Denote $S = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$
Now regarding $aS$ as a polynomial in $x$ and going through the process of completing the square, we get
$aS = (ax + hy + gx)^2 + aby^2 + acz^2 + 2afyz - (gz + hy)^2$
$\therefore aS = (ax + hy + gx)^2 + (ab - h^2)y^2 - 2(gh - af)yz + (ac - g^2)z^2$
$\therefore aS = (ax + hy + gx)^2 + (Cy^2 - 2Fyz + Bz^2)$,
where $C = ab - h^2, F = gh - af, B = ac - g^2$.
$\therefore BC - F^2 = a(abc + 2fgh - af^2 - bg^2 - ch^2)$.
As, $(abc + 2fgh - af^2 - bg^2 - ch^2) = 0, \therefore BC - F^2 = 0$. Hence for some $r$ and $t$, $Cy^2 - 2Fyz + Bz^2 = -(ry + tz)^2$.
Therefore $aS$ can be expressed as a difference of two squares and so has linear factors. Thus, if $a \neq 0$ and if $(abc + 2fgh - af^2 - bg^2 - ch^2) = 0$, then the given equation (22) represents two planes.
If $a = 0$ but $b \neq 0$ or $c \neq 0$, there is a similar argument. If $a = b = c = 0$ then $2fgh = 0$. It gives at least one of $f, g, h = 0$; this is sufficient for $2fyz + 2gzx + 2hxy$ to factorize.

**Remark 6.12** *If $\theta$ is the angle between the planes represented by the equation (22); and if $ll' + mm' + nn' \neq 0$, then*
$\tan\theta = \dfrac{\sqrt{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2}}{ll' + mm' + nn'}$
$\therefore \tan\theta = \dfrac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c}$.



*The planes will be at right angles if* $a + b + c = 0$.

**Example 6.4** Show that the equation $12x^2 - 2y^2 - 6z^2 - 2xy + 7yz + 6zx = 0$ represents a pair of planes. Also find the angle between them.
**Solution.** Comparing $12x^2 - 2y^2 - 6z^2 - 2xy + 7yz + 6zx = 0$ with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$,
we have $a = 12, b = (-2), c = (-6), 2h = (-20), 2f = 7, 2g = 6$.
$\therefore abc + 2fgh - af^2 - bg^2 - ch^2$
$= 12(-2)(-6) + 2(\frac{7}{2})(3)(-1) - 12(\frac{7}{2})^2 - (-2)3^2 - (-6)(-1) = 0$.
Hence, the given second degree equation represents a pair of planes.
Let $\theta$ be the angle between two planes. Then
$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c} = \frac{2\sqrt{\frac{7}{2}^2+3^2+(-1)^2-12(-2)(-6)-(-6)(12)}}{12+(-2)+(-6)}$
$\therefore \tan\theta = \frac{5\sqrt{17}}{4}$. It gives, $\cos\theta = \frac{4}{21}$. Thus $\theta = \cos^{-1}(\frac{4}{21})$.

## 6.13 Illustrative Examples

**Example 6.5** Find the equation of the plane passing through the intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and the point $(1, 1, 1)$.
**Solution.** The required plane passes through the line of intersection of the given planes. Therefore its equation is of the form

$$(2x + 3y + 4z + 5) + k(x + y + z - 6) = 0 \text{ for some } k \in \mathbb{R}.$$

Also it is given that the plane passes through the point$(1, 1, 1)$. Coordinates of these point satisfy the equation $(5.12.1)$, we get $k = \frac{14}{3}$. Substituting $k = \frac{14}{3}$, we get equation of the required plane as $20x + 23y + 26z - 69 = 0$.

**Example 6.6** Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0, 2x + y - z + 5 = 0$.
**Solution.** The required plane passes through the line of intersection of the given planes. Therefore its equation is of the formed

$$(x + 2y + 3z - 4) + k(2x + y - z + 5) = 0 for\ some\ k \in \mathbb{R} \qquad (23)$$

$d.r.s$ of the normal to the plane (6.23) are $1 + 2k, 12 + k, 13 - k$. The required plane is perpendicular to the plane $5x+3y+6z+8 = 0$. Therefore $5(1+2k)+3(2+k)+6(3-k) = 0$. From this equation, we get $k = \frac{-29}{7}$. Substitute $k = \frac{-29}{7}$in $(5.12.2)$ we get the required equation of the plane as $51x + 15y - 50z + 173 = 0$.

**Example 6.7** Find the equations to the planes through the line of intersection of the planes $x + 2y + 2z - 4 = 0, 2x + y - z + 5 = 0$ and
$(a)$ parallel to $x$-axis $(b)$ parallel to $y$-axis and $(c)$ parallel to $z$-axis.
**Solution.** The required equation of the plane passes through the line of intersection if the given planes. Therefore its equation is of the form

$$(x + 2y + 2z - 4) + k(2x + y - z + 5) = 0 for\ some\ k \in \mathbb{R} \qquad (24)$$

$d.r.s$ of the normal to the plane (6.24) are $1 + 2k, 2 + k, 2 - k$
$(a)$ $d.r.s$ of the $x-axis$ are $1, 0, 0$. The plane (6.24) is parallel to $x-axis$. Hence normal to plane (6.24) is perpendicular to $x - axis$. $1(1 + 2k) + 0(2 + k) + 0(2 - k)$. It gives $k = \frac{-1}{2}$. Substitute this value $k$ in (6.24) we get the equation of the plane parallel to $x - axis$ as $3y + 5z - 13 = 0$. $(b)$ By the similar argument, the equation of the plane parallel to $y - axis$ is $3x - 4y + 14 = 0$. $(c)$ The equation of the plane parallel to $z - axis$ is $5x + 4y + 6 = 0$.

**Example 6.8** Show that the distance between the parallel planes $2x-2y+z+3 = 0$ and $4x - 4y + 2z + 5 = 0$ is $\frac{1}{6}$.
**Solution.** The distance between two parallel planes is the distance of any one point from one plane to other. The point $P(1, 1, -3)$ lies on the plane $2x - 2y + z + 3 = 0$. Hence the distance between two parallel planes = perpendicular distance from $P$ to the plane $4x - 4y + 2z + 5 = 0$. The perpendicular distance equals $|\frac{4(1)-4(1)+2(-3)+5}{\sqrt{42+(-4)^2+2^2}}| = \frac{1}{6}$. Thus the distance between given two parallel planes is $\frac{1}{6}$.

**Example 6.9** A variable plane which remains at a constant distance $3p$ from the origin, the cuts to coordinate axes at $A, B\ and\ C$ Show that the locus of the centroid of the $\triangle ABC$ is $x^2 + y^2 + z^2 = p^2$.



**Solution.** Let the equation of the variable plane be $ax + by + cz + d = 0$, which is at a distance $3p$ from the origin. Then by the perpendicular distance formula,

$$|\frac{a(0) + b(0) + c(0) + d}{\sqrt{a^2 + b^2 + c^2}}| = 3 \text{ i. e. } \frac{d^2}{a^2 + b^2 + c^2} = 9p^2.$$

Suppose the plane meets the $x, y$ and $z$-axis at $A, B$ and $C$ respectively. Then, we have $A(\frac{-d}{a}, 0, 0), B(0, \frac{-d}{b}, 0), C(0, 0, \frac{-d}{c})$. Let $G(x_1, y_1, z_1)$ be the centroid of the $\triangle ABC$. Then

$$x_1 = \frac{\frac{-d}{a} + 0 + 0}{3}, y_1 = \frac{0 + \frac{-d}{b} + 0}{3}, z_1 = \frac{0 + 0 + \frac{-d}{c}}{3}.$$

Squaring and adding these equations, we get

$$x_1^2 + y_1^2 + z_1^2 = \frac{d^2}{9(a^2 + b^2 + c^2)} = p^2.$$

Hence the locus of the centroid of the $\triangle ABC$ is $x^2 + y^2 + z^2 = p^2$.

Or

Let the equation of the variable plane be $lx + my + nz = 3p$. Suppose the plane meets the $x, y$ and $z$-axis at $A, B$ and $C$ respectively. Then, we have $A(\frac{3p}{l}, 0, 0)$, $B(0, \frac{3p}{b}, 0)$, $C(0, 0, \frac{3p}{n})$. Let $G(x_1, y_1, z_1)$ be the centroid of the $\triangle ABC$. Then $x_1 = \dfrac{p}{l}, y_1 = \dfrac{p}{m}, z_1 = \dfrac{p}{n}$. Squaring and adding these equations, we get $x_1^2 + y_1^2 + z_1^2 = p^2$. Hence the locus of the centroid of the $\triangle ABC$ is $x^2 + y^2 + z^2 = p^2$.

**Example 6.10** The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle $\alpha$. Prove that the equation of the plane in its new position is $(lx + my \pm \sqrt{l^2 + m^2} \tan \alpha)z = 0$.
**Solution.** Let $\pi$ be the plane obtained by rotating the plane $lx + my = 0$ about its line of intersection with the plane $z = 0$ through an angle $\alpha$. Then the equation of the plane $\pi$ is of the form

$$lx + my + kz = 0\ for\ some\ k \in \mathbb{R}. \qquad (25)$$

Note that angle between the plane $\pi$ and the plane $lx+my=0$ is $\alpha$. Using angle between two planes,we get $\cos\alpha = \frac{l^2+m^2}{\sqrt{l^2+m^2+k^2}\sqrt{l^2+m^2}}$
$\cos^2\alpha = \frac{(l^2+m^2)^2}{(l^2+m^2+k^2)(l^2+m^2)}$.
From this equation,we get $k = \pm\sqrt{l^2+m^2}\tan\alpha$. Substitute this value of $k$ in (6.25), we have $(lx+my\pm\sqrt{l^2+m^2}\tan\alpha)z = 0$.

**Example 6.11** Find the locus of a point which is equidistant from the two planes $x+2y+2z=3$ and $3x+4y+12z+1=0$.
**Solution.** Let $P(x_1,y_1,z_1)$ be a point which is equidistant from the given two planes.Then by the perpendicular distance formula,we have

$$\begin{aligned} \left|\frac{x_1+2y_1+2z_1-3}{\sqrt{1^2+2^2+2^2}}\right| &= \left|\frac{3x_1+4y_1+12z_1+1}{\sqrt{3^2+4^2+12^2}}\right| \\ \therefore \frac{x_1+2y_1+2z_1-3}{3} &= \pm\frac{3x_1+4y_1+12z_1+1}{13} \\ \therefore 2x_1+7y_1-5z_1-21=0; & \quad 11x_1+19y_1+13z_1-18=0. \end{aligned}$$

Hence the locus of a point which is equidistant from the given planes is $2x+7y-5z-21=0$ *or* $11x+19y+13z-18=0$.

**Example 6.12** Find the joint equation of the planes $2x+3y-z=0$ and $x-y+5z=0$.
**Solution.** The joints equation of given two planes is $(2x+3y-z)(x-y+5z)=0$. On simplification, we get $2x^2-y^2-5z^2+16yz+9zx+xy=0$

## 6.14 Exercise

1. Find the equation of the plane passing through the point $(2,3,5)$ and perpendicular to the line whose d.r.s. are 3,-2,6.

2. Find the equation of the plane passing through the point $(1,-3,-4)$ and parallel to the plane $6x+2y-3z=5$.

3. Find the distance from the point P to the plane $\pi$, where

   (a) $\pi$ is $2x+y-z=4$, P is $(2,3,5)$



   (b) $\pi$ is $4x-3y-z=4$, P is $(4,2,3)$

   (c) $\pi$ is $5x-3y+2z=6$, P is $(3,-1,2)$

4. Find the locus of a point the sum of the squares of whose distances from the planes $x+y+z=0$, $x-z=0$ and $x-2y+z=0$ is $9$.

5. Show that the equation $x^2-y^2+2z^2+yz+3zx+x+y+z=0$ represents a pair of planes. Also find the angle between the planes.

6. Show that the points $(-2,2,-1)$ and $(1,-1,1)$ lie on different sides of the plane $x-2y+z+5=0$.

7. Find the equations of the planes bisecting the angles between the planes $x+2y+2z-3=0$ and $3x+4y+12z+1=0$ and specify the one which bisects the acute angle.

8. Find the equations of planes parallel to the plane $x-2y+2z=3$ whose perpendicular distance from the point $(1,2,3)$ is 1.

9. Find the perpendicular distance between the parallel planes $2x-2y+z+6=0$ and $4x-4y+2z+5=0$.

10. Find the equation of the plane passing through the point $(4,0,-1)$ and parallel to the plane $2x-5y+\sqrt{7}z+5=0$.

11. Find the equation of the plane passing through the point $(1,2,1)$ and containing the $y$-axis.

12. Find the equation of the plane passing through the line of intersection of the planes $2x+y-z+5=0=x+2y+2z-4$ and is perpendicular to the plane $5x+3y+6z+11=0$.

13. Find the equation of the plane through the point $(3,3,1)$ and perpendicular to the line joining the points $(2,-1,3)$ and $(4,2,-1)$.

14. Find the equation of the plane passing through the line of intersection of the planes $2x+y-z=3$ and $5x-3y+4z+9=0$ and is parallel to the line whose d.r.s.are $2,4,5$.

15. Write the equations of the following planes:
    (a) parallel to the $XZ$-plane through $(2, 4, 5)$
    (b) parallel to the $XY$-plane and 5 units from it.

16. Find the locus of a point which is always at a distance $\frac{7}{\sqrt{3}}$ units from the plane $x + y + z + 1 = 0$.

17. Find the locus of a point whose distance from the origin is 7 times its distance from the plane $2x + 3y - 6z = 2$.

18. Find the equation of the plane passing through the line of intersection of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and passing through the origin.

19. The plane $x - 2y + 3z = 0$ is rotated through a right angle about its line of intersection with the plane $2x+3y-4z+5 = 0$. Show that the equation of the plane in its new position is $22x + 5y - 4z + 35 = 0$.

20. From the point $P(a, b, c)$ perpendiculars $PM$ and $PN$ are drawn to the $ZX$-plane and $XY$-plane. Find the equation of the plane $OMN$, where $O$ is the origin.

21. Find the perpendicular bisecting plane of the segment $(2, 5, -3)$ and $(0, -4, 2)$.

22. Find the equation of a plane which bisects the acute angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$.

23. Determine whether the following points lie on the same side of the plane $3x - 2y + 4z = 10$ : $(1, -1, 2)$, $(0, 1, 1)$ and $(0, 0, 2)$.

24. Find the equation of a plane which bisects the angle between the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z = 3$ which contains the origin.

25. Determine which of the following equations represent pairs of planes. Also, find the angle between the pair if the equation represents a pair of plane.



    (a) $3x^2 - 10yz + 5zx - 6xy + x - 2y = 0$
    (b) $yz + zx + 2xy + 2x + 5 = 0$
    (c) $x^2 + y^2 + z^2 + 4yz + 2zx + 2xy + x - 2y + 1 = 0$
    (d) $2x^2 - 2y^2 - 3z^2 + 5yz - 5zx + 3xy + 4x - 7y + 9z - 6 = 0$.

26. Show that the plane $14x - 8y + 13 = 0$ bisects the obtuse angle between the planes $3x + 4y - 5z + 1 = 0$ and $5x + 12y - 13z = 0$.

27. Show that the planes $3x - 5y + 2 = 0$, $6x + y - 2z = 13$ and $11y - 2z = 17$ pass through one line.

28. Find the condition that the planes $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ may pass through one line.

## 6.15 Answers

(1) $3x - 2y + 6z = 30$ (2) $6x + 2y - 3z = 12$ (3) $(a)$ $\frac{\sqrt{6}}{3}$ $(b)$ $\frac{3\sqrt{7}}{14}$
$(c)$ $\frac{8\sqrt{38}}{19}$ (4) $3x^2 + 3y^2 + 3z^2 + 2xz - 27 = 0$ (5) $\theta = \cos^{-1}(\frac{\sqrt{2}}{3})$
(7) $2x+7y-5z = 21$;$11x+19y+31z = 18$ which bisects the acute angle
(8) $x-2y+2z = 6$; $x-2y+2z = 0$ (9) $\frac{7}{6}$ (10) $2x-5y+\sqrt{7}z = 8-\sqrt{7}$
(11) $x - z = 0$ (12) $51x + 15y - 50z + 173 = 0$
(13) $2x + 3y - 4z = 11$ (14) $7x + 9y - 10z = 27$
(15) $(a)$ $y = 4$ $(b)$ $z \pm 5$ (16) $x + y + z = 6$;$x + y + z + 8 = 0$
(17) $3x^2 + 8y^2 + 35z^2 - 36yz - 24zx + 12xy - 8x - 12y + 24z + 4 = 0$
(18) $3x + 2y + z = 0$ (19) $bcx - acy - abz = 0$ (21) $2x + 9y - 5z = 9$
(22) $23x - 13y - 32z + 45 = 0$ (23) All points lie on the same side.
(24) $67x - 162y + 47z + 44 = 0$
(25) $(a)$ and $(d)$ represent pair of planes. Angles are $\cos^{-1}\frac{3}{\sqrt{170}}$ and $\cos^{-1}\frac{-3}{\sqrt{70}}$.
(28) $a^2 + b^2 + c^2 + 2abc = 1$. <u>*Hint*</u>: For some $\lambda$ and $\mu$, we have
$(x - cy - bz) + \lambda(y - cx - az) = \mu(z - bx - ay)$.

# Chapter 7

# Lines in 3D

## 7.1 Introduction:

The reader has already been introduced to the study of lines in three dimensions. In this chapter we shall study the condition that two given lines are coplanar and skew lines. Note that in the Section 7.2 we take the revision of the known results.

## 7.2 Equations of a Line:

We know that the intersection of two planes is a line. Consider two intersecting planes

$$a_1x + b_1y + c_1z + d_1 = 0 \quad (1)$$
$$a_2x + b_2y + c_2z + d_2 = 0 \quad (2)$$

Let $L$ be the line of intersection of the planes given by (1) and (2).
Any point in $L$ is common to both the planes. Hence its coordinates will satisfy the equations (1) and (2). Also if the co-ordinates of a point satisfy both equations (1) and (2), then it lies in the line $L$. Hence two equations (1) and (2) taken together are the equations of the line $L$. There equations are said to be the general equations of a line. Thus, a straight line is represented by two equations of the first degree in $x, y, z$.

### 7.2.1 Symmetrical form of the Equations of a Line:

Equations of a line passing through the given point $A(x_1, y_1, z_1)$ and having direction cosines $l, m, n$ are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (3)$$





Each ratio equals $\frac{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}{\sqrt{l^2+m^2+n^2}} = t$ (say). Thus, $|t|$ gives the distance between the point $(x, y, z)$ and the point $(x_1, y_1, z_1)$ and $t$ is called as directed distance from $(x_1, y_1, z_1)$ to $(x, y, z)$. Observe that we can rewrite (3) as

$$x = x_1 + lt, y = y_1 + mt, z = z_1 + nt \quad (4)$$

Equations given by (4) represent the co-ordinates of any point on the line at a distance $t$ from the point $A(x_1, y_1, z_1)$. equations given by (4) are called parametric equations of a straight line.

**Remark 7.1** From the equations given by (4), we observe that $t$ is the actual distance of the point $P(x, y, z)$ from the given point $A(x_1, y_1, z_1)$, becomes $l, m, n$ are d.c.s. of the line $(l^2 + m^2 + n^2 = 1)$.

However, if $l, m, n$ are given to be proportional to the direction cosines of $AP$, then $t$ will be proportional to the distance $AP$. Thus if instead of the direction cosines $l, m, n$ the direction ratios $a, b, c$ are given the equations of the line are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

This form is known as symmetrical form of equations of a line.

### 7.2.2 Equations of a Line Passing Through Two Points

Let the line $L$ pass through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Let $P(x, y, z)$ be any point on the line $L$. The direction ratios of $AP$ are $x - x_1, y - y_1, z - z_1$, and of $AB$ are $x_1 - x_2, y_1 - y_2, z_1 - z_2$. Since the line $AP$ is parallel to $AB$, d.r.s of $AP$ are proportional to the d.r.s. of $AB$.

$$\therefore \frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{z_1 - z_2} = \frac{z - z_1}{z_1 - z_2}$$

are the required equations of the line $L$ passing through the points $A(x_1, y_1, z_1)$ and $B(z_2, y_2, z_2)$.

### 7.2.3 Transformation of the equations of a line from the asymmetric form to the symmetric form.

Let the equations of a line $L$ in asymmetrical form be

$$a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0.$$

To transform this asymmetrical form to the symmetrical form, we must know the coordinates of a point on the line $L$ are d.r.s. of the line.

The following illustrative example will make the procedure clear.

**Example 7.1** Find the symmetric form of the equations of the line

$$x + y + z + 1 = 0; 4x + y - 2z + 2 = 0.$$

**Solution:** Let $L$ be a line whose equations are given by

$$x + y + z + 1 = 0; 4x + y - 2z + 2 = 0 \qquad (i)$$

To transform the equations of the line into the symmetric form, first we find co-ordinates of a point on the line $L$. We may consider, for the sake of convenience, the intersection of the line with any one of the co-ordinate planes, say $z = 0$, so that

$$x + y + 1 = 0; 4x + y + 2 = 0 \qquad (ii)$$

Solving equations given by (ii) simultaneously, we get the required point as $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$ . Next, let $l, m, n$ be the d.c.s. of the line $L$. The line $L$ lies in both the planes given by (i). Hence it is perpendicular to the normals of the two planes; and as d.r.s. of the normals to the planes $x+y+z+1 = 0$ and $4x + y - 2z + 2 = 0$ are $1, 1, 1$ and $4, 1, -2$.

$$\begin{aligned} \therefore l + m + n &= 0 \\ 4l + m - 2n &= 0 \end{aligned}$$

by cramer's rule, solving these equations for $l, m, n$ we have

$$\frac{l}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-m}{\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix}}$$



$$\therefore \frac{l}{-3} = \frac{m}{6} = \frac{n}{-3}; \text{ i.e.; } \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}.$$

So that d.r.s. of the required line are $1, -2, 1$. Hence the equations of the line $L$ in symmetric form are $\dfrac{x+\frac{1}{3}}{1} = \dfrac{y+\frac{2}{3}}{-2} = \dfrac{z}{1}$.

### 7.2.4 Angle between a line and a plane:

To find the angle between the line

$$L : \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

and the plane $\alpha : ax + by + cz + d = 0$. If $\theta$ is the angle between the line $L$ and the plane $\alpha$, then the angle between the line $L$ and the normal to the plane $\alpha$ is $\frac{\pi}{2} - \theta$ (see fig. 7.1).

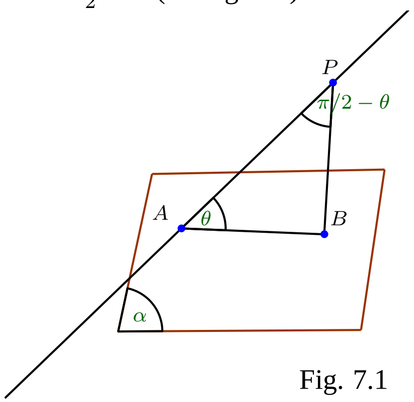



Fig. 7.1

The d.r.s. of the line $L$ are $l, m, n$ while d.r.s. of the normal (say $PB$) are $a, b, c$. Hence,

$$\begin{aligned} \cos\left(\frac{\pi}{2} - \theta\right) &= \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{l^2 + m^2 + n^2}} \\ \text{i. e. } \sin\theta &= \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}} \text{ as } l^2 + m^2 + n^2 = 1. \end{aligned}$$

The line will be parallel to the plane if $\theta = 0$, i.e. if $al + bm + cn = 0$.

## Illustrative Examples

**Example 7.2** Find the equations of a line through $(-2, 3, 4)$ and parallel to the planes $2x + 3y + 4z = 5$ and $3x + 4y + 5z = 6$.

**Solution:** Let $L$ be a line passing through the point $A(-2, 3, 4)$ and parallel to the planes

$$\alpha_1 : 2x + 3y + 4z = 5 \text{ and } \alpha_2 : 3x + 4y + 5z = 6.$$

d.r.s. of the normal to the plane $\alpha_1$ are $2, 3, 4$; and
d.r.s. of the normal to the plane $\alpha_2$ are $3, 4, 5$.
Given that the line $L$ is parallel to the planes $\alpha_1$ and $\alpha_2$, hence $L$ is perpendicular to their normals, we get $2a + 3b + 4c = 0$ and $3a + 4b + 5c = 0$, where $a, b, c$ are d.r.s. of the line $L$. Solving there equations for $a, b, c$ we get $\frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$. Therefore d.r.s of the line $L$ are $1, -2, 1$. Hence its equations are $\dfrac{x+2}{1} = \dfrac{y-3}{-2} = \dfrac{z-4}{1}$.

**Example 7.3** Find the equations of a line joining the points $(-2, 1, 3)$ and $(3, 1, -2)$.

**Solution:** Let $L$ be a line passing through the points $A(-2, 1, 3)$ and $B(3, 1, -2)$. The d.r.s. of the line $L$ are $3 - (-2), 1 - 1, -2 - 3$; i.e., $5, 0, -5$; i.e., $1, 0, -1$. Equations of the line $L$ are

$$\frac{x+2}{1} = \frac{y-1}{0} = \frac{z-3}{-1}.$$

**Example 7.4** Find the equations of the line through $(3, 1, 2)$ and perpendicular to the plane $2x - 2y + z + 3 = 0$. Also find the coordinates of the foot of the perpendicular.

**Solution:** Let $L$ be a line passing through the point $(A(3, 1, 2)$ and perpendicular to the plane $\alpha : 2x - 2y + z + 3 = 0$. d.r.s. of the normal to the plane $\alpha$ are $2, -2, 1$. The line $L$ is perpendicular to the plane $\alpha$. d.r.s. of $L$ are $2, -2, 1$. Equations of the line $L$ are

$$\frac{x-3}{2} = \frac{y-1}{-2} = \frac{z-2}{1} = t \text{ (say)}.$$



Coordinates of any point on the line $L$ are $(3 + 2t, 1 - 2t, 2 + t)$. For some $t$, these are the coordinates of the foot of the perpendicular. Hence $(3 + 2t, 1 - 2t, 2 + t)$ lies on the plane $\alpha$.

$$2(3 + 2t) - 2(1 - 2t) + (2 + t) + 3 = 0 \text{ i.e.} 9t + 9 = 0.$$

Hence, $t = -1$ and the coordinates of the foot of the perpendicular are $(1, 3, 1)$.

**Example 7.5** Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{6}$.

**Solution:** Let $A(1, -2, 3)$ and let the line $AB$ be drawn parallel to the line $L : \frac{x}{2} = \frac{y}{3} = \frac{z}{6}$, so as to intersect the plane $\alpha : x - y + z = 5$ at the point $B$. (see Fig. 7.2).

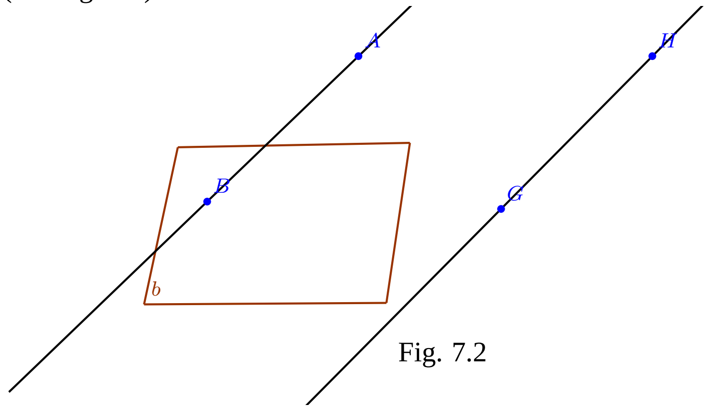



Fig. 7.2

As the line $AB$ is parallel to the line $L$, d.r.s. of $AB$ are $2, 3, 6$. Equations of the line $L$ are

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6} = t \text{ (say)}$$

Coordinates of any point on the line $AB$ are $(1 + 2t, -2 + 3t, 3 + 6t)$. For some $t$, these are the coordinates of $B$. But $B$ lies on the plane $\alpha$.

$$\therefore (1 + 2t) - (-2 + 3t) + (3 + 6t) = 5 \therefore t = -\frac{1}{5}.$$

Coordinates of $B$ are $\left(\frac{3}{5}, \frac{-13}{5}, \frac{9}{5}\right)$ . Hence, the required distance is

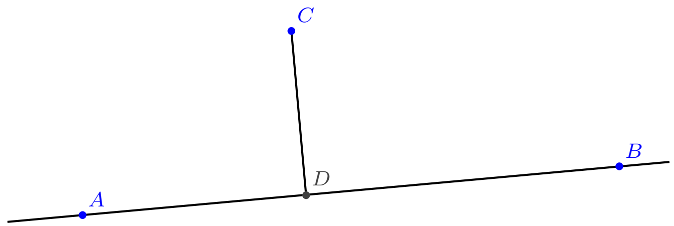
$$AB = \sqrt{\left(1-\frac{3}{5}\right)^2 + \left(-2+\frac{13}{5}\right)^2 + \left(3-\frac{9}{5}\right)^2} = \frac{7}{5}.$$

**Example 7.6** Find the length of the perpendicular drawn from the point $(5, 4-1)$ to the line $\frac{x-1}{2} = \frac{y}{9} = \frac{z}{5}$.

**Solution:** Let $A(5, 4, -1)$; and let $B$ be the foot of the perpendicular from $A$ to the line

$$L : \frac{x-1}{2} = \frac{y}{9} = \frac{z}{5} = t \text{ (say)}$$

(see Fig. 7.3)




Coordinates of any point on the line $L$ are $(1+2t, 9t, 5t)$. $B$ lies on the line $L$, for some $t$ these are the coordinates of $B$. Now d.r.s. of $AB$ are $1+2t) - 5, 9t-4, 5t+1$, i.e. $2t-4, 9t-4, 5t+1$. d.r.s. of $L$ are $2, 9, 5$. $AB$ is perpendicular to $L$.

$$\therefore 2(2t-4) + 9(9t-4) + 5(5t+1) = 0.$$

$$\therefore t = \frac{39}{110}. \quad B\left(\frac{188}{110}, \frac{351}{110}, \frac{195}{110}\right)$$

Required length of the perpendicular,

$$AB = \sqrt{\left(\frac{188}{110}-5\right)^2 + \left(\frac{351}{110}-4\right)^2 + \left(\frac{195}{110}+1\right)^2} = \frac{\sqrt{231990}}{110}$$



**Example 7.7** Show that the line $\dfrac{x-7}{4} = \dfrac{y-5}{3} = \dfrac{z-3}{2}$ intersects the line $5x-3y+z-10 = 0; 2x+7y-4z-16 = 0$. Also find the coordinates of the point of intersection.

**Solution:** Let $L_1 : \dfrac{x-7}{4} = \dfrac{y-5}{3} = \dfrac{z-3}{2}$; and

$$L_2 : 5x - 3y + 3 - 10 = 0;\ 2x + 7y - 4z - 16 = 0.$$

Coordinates of any point on the line $L$ are $(7+4t, 5+3t, 3+2t)$. Suppose the line $L_1$ intersects the plane $5x-3y+z-10 = 0$ at $P$.
For some $t$, let the coordinates of $P$ be $(7+4t, 5+3t, 3+2t)$. But $P$ lies on the plane $5x-3y+z-10 = 0$. Therefore $5(7+4t) - 3(5+3t) + (3+2t) - 10 = 0$. From this equation, we get $t = -1$. Therefore, we get $P$ as $(3, 2, 1)$. If the coordinates of $P$ satisfy the equation of another plane $2x+7y-4z-16 = 0$, then we say that the lines $L_1$ and $L_2$ intersect.
Now $2(3) + 7(2) - 4(1) - 16 = 0$.
Coordinates of $P$ satisfy the equation of the plane $2x+7y-4z-16 = 0$. Hence the lines $L_1$ and $L_2$ intersect; and the coordinates of the point of intersection are $(3, 2, 1)$.

**Example 7.8** Find the equation of the plane containing point $(0, 7, -7)$ and the line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$.
**Solution:** Let $\alpha$ be a plane containing the point $A(0.7, -7)$ and the line

$$L : \frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}.$$

Note that the point $B(-1, 3, -2)$ lies on the line, and hence lies on the plane $\alpha$. d.r.s. of the line $AB$ are $0-(-1), 7-3, -7-(-2)$; i.e; $1, 4, -5$ and d.r.s. of the line $L$ are $-3, 2, 1$. Let $a, b, c$ be d.r.s. of the normal to the plane $\alpha$. Line $AB$ and $L$ lie on the plane $\alpha$, therefore both the lines are perpendicular to the normal. Therefore, $a+4b-5c = 0$ and $-3a+2b+c = 0$. Solving these equations for $a, b$ and $c$.

$$\frac{a}{14} = \frac{-b}{-14} = \frac{c}{14};\ \text{i.e. } \frac{a}{1} = \frac{b}{1} = \frac{c}{1}.$$

Now, d.r.s. of the normal to the plane $\alpha$ are $1, 1, 1$. The required equation of the plane $\alpha$ is $1(x-0) + 1(y-7) + 1(z+7) = 0$; i.e. $x+y+z = 0$.

## 7.3 Coplanar Lines:

### 7.3.1 Condition for a line to lie in a plane:

To find the conditions for the line $L: \dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ to lie in the plane $\alpha: \quad ax+by+cz+d=0$.

The line $L$ lies in the plane $\alpha$ if and only if every point on the line lies in the plane $\alpha$. Now, the coordinates of any point on the line $L$ are $(x_1+lt, y_1+mt, z_1+nt)$. These coordinates satisfy the equation of the plane $\alpha$.

$$\begin{aligned} \therefore a(x_1+lt)+b(y_1+mt)+c(z_1+nt)+d &= 0 \\ \therefore (ax_1+by_1+cz_1+d)+(al+bm+cn)t &= 0 \end{aligned}$$

This equation is true for every value of $t$. This is possible if and only if $ax_1+by_1+cz_1+d=0$ and $al+bm+cn=0$; which are the required two conditions. Hence a line will lie in the plane if and only if the point $(x_1,y_1,z_1)$ lies in the plane and the normal is perpendicular to the line.

**Remark 7.3.1** From the conditions for a line to lies on a plane, it is easy to see that the general equation of a plane containing the line

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, where $al+bm+cn=0$.

**Example 7.9** Show that the line $x+10 = \dfrac{8-y}{2} = z$ lies in the plane $x+2y+3z=6$.

**Solution :** The parametric equation of the line $L$ is $x=t-10, y=8-2t, z=t$. Hence, the general point on line $L$ is given by $(t-10, 8-2t, t)$. Note that it satisfies the equation of the plane $x+2y+3z=6$. Thus, the line $x+10 = \dfrac{8-y}{2} = z$ lies in the plane $x+2y+3z=6$.

or



**Solution :** Let $L: x+10 = \frac{8-y}{2} = z$. The point $A(-10,8,0)$ lies on the line $L$ and d.r.s. of $L$ are $l=1, m=-2, n=1$. d.r.s. of the normal to the plane $x+2y+3z=6$ are $a=1, b=2, c=3$. To show that the line $L$ lies in the plane for we show that $A$ lies on the plane and $al+bm+cn=0$. Consider $-10+2\times 8+2\times 0=6$. Coordinates of $A$ satisfy the equation of the plane $x+2y+3z=6$. Also $al+bm+cn = 1\times 1+2\times(-2)+3\times 1=0$. Hence the line $x+10 = \frac{8-y}{2} = z$ lies in the plane $x+2y+3z=6$.

### 7.3.2 Condition for two lines to be coplanar:

To find the condition that the two given lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \qquad (i)$$
$$\frac{x-x_2}{l_2} = \frac{y-y_1}{m_2} = \frac{z-z_1}{n_2} \qquad (ii)$$

are coplanar. By Remark 6.3.1 a plane containing the line (i) will be of the form

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0; \qquad (iii)$$

where $a,b,c$ being numbers not all zero simultaneously satisfying the condition

$$al_1+bm_1+cn_1=0 \qquad (iv)$$

The line (ii) will lie in the plane (iii) if and only if

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0 \qquad (v)$$

subject to $\quad al_2+bm_2+cn_2=0 \qquad (vi)$

Eliminating $a,b,c$ from (iv), (v) and (vi), we get

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

as the required condition for the lines to be coplanar. When this condition of coplanarity is satisfied, the equation of the plane containing the lines (i) and (ii) is given by $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

**Example 7.10** Show that the lines

$$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3} \text{ and } \frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$$

are coplanar and find the equation of the plane containing them.

**Solution.** Let

$$L_1 : \frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3} \text{ and } L_2 : \frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}.$$

The points $(x_1, y_1, z_1) = (-3, -5, 7)$ and $(x_2, y_2, z_2) = (-1, -1, -1)$ lie on the lines $L_1$ and $L_2$ respectively. d.r.s of $L_1$ are $l_1 = 2, m_1 = 3, n_1 = 3$; and that of $L_2$ are $l_2 = 4, m_2 = 5, n_2 = -1$. Consider

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 2 & 4 & -8 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 4 & -8 \\ 0 & -1 & 5 \\ 0 & -3 & 15 \end{vmatrix}$$

by performing $R_2 - R_1$ and $R_3 - 2R_2$. Since, the third row is 3 times the second row, the determinant is zero. Hence the lines $L_1$ and $L_2$ are coplanar. The equation of the plane containing them is
$\begin{vmatrix} x+3 & y+5 & z-7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0$ which on expansion gives $6x - 5y - z = 0$.

## 7.4 Sets of conditions which determine a line

### 7.4.1 Number of arbitrary constants in the equations of a straight line

Let $L$ be a line whose equations are $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$. These are equivalent to $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m}$; $\dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$. From these equations, we get $x = \dfrac{l}{m}y + \dfrac{mx_1 - ly_1}{m}$ and $y = \dfrac{m}{n}z + \dfrac{ny_1 - mz_1}{n}$. These equations contain $\dfrac{l}{m}, \dfrac{m}{n}, x_1 - \dfrac{l}{m}y_1$ and $y_1 - \dfrac{m}{n}z_1$ as the five arbitrary



constants, viz . $\dfrac{l}{m}, \dfrac{m}{n}, x_1, y_1, z_1$. Hence the equations of a line involve five arbitrary constants.

### 7.4.2 Sets of conditions which determine a line:

We know that the equations of a straight line involve four arbitrary constants and as such any four geometrical conditions.
We have seen equations of a line using following conditions:

1. A line through a given point and a given line direction.
2. A line passing through two given points.
3. A line through a given point and parallel to the two given planes.
4. A line through a point and perpendicular to the two given lines.
5. A line through a given point and intersecting two given lines.
6. A line intersecting two given lines and a given direction.
7. A line intersecting two given lines at right angles.
8. A line intersecting a given line at right angles and passing through a given point.

## 7.5 Skew lines and shortest distance

**Definition 7.1** *Two lines are said to be* ***non coplanar or skew*** *lines if no plane can be drawn to contain both of them.*

Therefore such lines are neither parallel nor intersecting. A line is completely determined if it intersects two lines at right angles. Thus, there is one and only one line which intersects the two given skew lines at right angles and the segment of this line terminated by the two skew lines is known as the ***shortest distance*** between them.

### 7.5.1 To find the length and the equations of the line of shortest distance between two lines

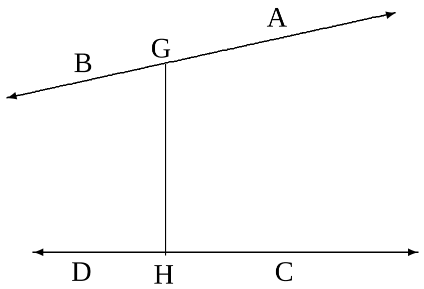



Figure 7.4

Let the equations of the two skew lines $AB$ and $CD$ be

$$AB: \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \tag{5}$$

$$CD: \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \tag{6}$$

Let $GH$ be line which meets both skew lines $AB$ and $CD$ at right angles (see Fig. 7.4). Then $GH$ is the line of shortest distance between the lines $AB$ and $CD$; the length $GH$ being the magnitude. Let $l, m, n$ be the $d.c.s.$. Since $GH$ is perpendicular to both of $AB$ and $CD$, we have

.

$$ll_1 + mm_1 + nn_1 = 0 \text{ and } ll_2 + mm_2 + nn_2 = 0.$$

Solving these two equations, we determine $l, m, n$. The shortest distance $GH$ is the projection of $AC$ on $GH$. Hence

$$GH = l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2).$$

**A method to determine the equations of the line of shortest distance**
The line $GH$ intersects the lines $AB$ and $CD$. Therefore there is a unique plane determined by the lines $GH$ and $AB$, also there is a unique plane determined by the lines $GH$ and $CD$. The equation of the plane containing



the coplanar lines $GH$ and $AB$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \tag{7}$$

and the equation of the plane containing the coplanar lines $GH$ and $CD$ is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0. \tag{8}$$

**Example 7.11** Find the shortest distance and the equations of the line of shortest distance between the skew lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4}; \ \frac{x-3}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

**Solution.** We give two methods for the solution.
First Method. Let

$$L_1: \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4} \text{ and } L_2: \frac{x-3}{3} = \frac{y-4}{4} = \frac{z-5}{5}.$$

The $d.r.s.$ of $L_1$ and $L_2$ are 2,3,4 and 3,4,5 respectively. Let $L$ be the line of shortest distance between $L_1$ and $L_2$; and let $l, m, n$ be the $d, c, s,$ of the line $L$. $L$ is perpendicular to both $L_1$ and $L_2$.

$$2l + 3m + 4n = 0; 3l + 4m + 5n = 0$$

Solving these two equations for $l, m, n$, we get $\dfrac{l}{-1} = \dfrac{m}{2} = \dfrac{n}{-1}$. Thus, the $d.r.s.$ of the line $L$ are $-1, 2, -1$ and hence d.c.s. of $L$ are $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$. $A(1,2,4)$ and $B(2,4,5)$ lie on the lines $L_1$ and $L_2$ respectively. The length of the shortest distance between the lines $L_1$ and $L_2$ equals the length of the projection of the line segment $AB$ on the line $L$

$$\begin{aligned} &= (x_1 - x_2)l + (y_1 - y_2)m + (z_1 - z_2)n \\ &= (1-2)\frac{1}{\sqrt{6}} + (2-4)\frac{-2}{\sqrt{6}} + (4-5)\frac{1}{\sqrt{6}} = \frac{2}{\sqrt{6}}. \end{aligned}$$

The equation of the plane containing the lines $L$ and $L_1$ is

$$\begin{vmatrix} x-1 & y-2 & z-4 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 0; \text{ i.e., } 11x + 2y - 7z + 13 = 0.$$

The equation of the plane containing the lines $L$ and $L_2$ is

$$\begin{vmatrix} x-1 & y-2 & z-4 \\ 3 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 0; \text{ i. e. } 7x + y - 5z + 7 = 0.$$

Hence the equations of the line of shortest distance are

$$11x + 2y - 7z + 13 = 0 \text{ and } 7x + y - 5z + 7 = 0.$$

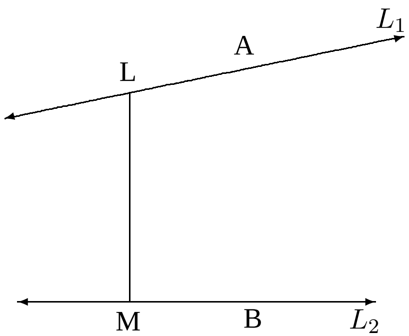



Figure 7.5

**Second Method.** Let

$$\begin{aligned} L_1 : \quad & \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4} = t \text{ and} \\ L_2 : \quad & \frac{x-3}{3} = \frac{y-4}{4} = \frac{z-5}{5} = s(say). \end{aligned}$$

$A(1, 2, 4)$ and $B(2, 4, 5)$ are points on the lines $L_1$ and $L_2$ respectively. Let $LM$ be the line of shortest distance between $L_1$ and $L_2$ (see Fig. 7.5).



Coordinates of any point on the line $L_1$ are $(1+2t, 2+3t, 4+4t)$. For some $t$, we get coordinates of $L$. Let $(2+3s, 4+4s, 5+5s)$ be the coordinates of $M$ for some $s$. Therefore $d.r.s.$ of $LM$ are $2t-3s-1, 3t-4s-2, 4t-5s-1$. $LM$ is perpendicular to both $L_1$ and $L_2$. It implies that

$$\begin{aligned} 2(2t-3s-1) + 3(3t-4s-2) + 4(4t-5s-1) &= 0 \\ \text{and } 3(2t-3s-1) + 4(3t-4s-2) + 5(4t-5s-1) &= 0 \end{aligned}$$

From these two equations, we get $29t-38s-12 = 0$ and $19t-25s-8 = 0$. Solving these two equations for $t$ and $s$, we have $t = \frac{-4}{3}$, $s = \frac{-4}{3}$. So that $L(\frac{-5}{3}, -2, \frac{-4}{3})$ and $M(-2, \frac{-4}{3}, \frac{-5}{3})$. By the distance formula, $LM = \frac{2}{\sqrt{6}}$. Equations of $LM$ are $\dfrac{3x+6}{1} = \dfrac{3y+4}{-2} = \dfrac{3z+5}{1}$.

### 7.5.2 Length of the perpendicular from a point to a line

To find the length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the given line $L : \dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$. Let $Q$ be the foot of the perpendicular from the point $P$ on the line $L$ (see Fig. 7.6). $HQ$ is projection of $HP$ on the line $L$. Hence, $HQ = l(x_1-\alpha) + m(y_1-\beta) + n(z_1-\gamma)$. By the distance formula, $HP^2 = (x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2$.

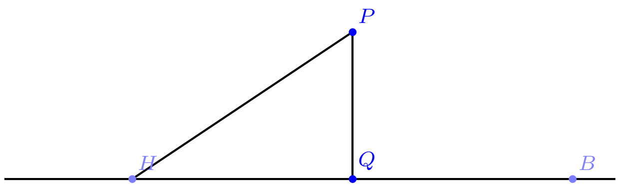



Fig. 7.6

We have $PQ^2 = HP^2 - HQ^2$. Thus, the length of the perpendicular form the point $P$ on the line $L$ is given by

$$\begin{aligned} PQ^2 &= (x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2 \\ &\quad -[l(x_1-\alpha) + m(y_1-\beta) + n(z_1-\gamma)]^2. \end{aligned}$$

**Example 7.12** Find the length of the perpendicular from the point $(4, -5, 3)$ to the line $\dfrac{x-5}{3} = \dfrac{y+2}{-4} = \dfrac{z-6}{5}$.

**Solution.** Let $L: \dfrac{x-5}{3} = \dfrac{y+2}{-4} = \dfrac{z-6}{5}$. $H(5, -2, 6)$ is a point on the line $L$, and $d.r.s.$ of the line $L$ are $3, -4, 5$. Therefore $d.c.s.$of the line $L$ are $\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}$. Let $Q$ be the foot of the perpendicular from $P(4, -5, 3)$ on the line $L$. $HQ$=projection of $HP$ on the line $L$.
$HQ = \frac{3}{5\sqrt{2}}(4-5) - \frac{4}{5\sqrt{2}}(-5+2) + \frac{5}{5\sqrt{2}}(3-6) = \frac{-6}{5\sqrt{2}}$. By the distance formula $HP = \sqrt{19}$. $PQ^2 = HP^2 - HQ^2 = 19 - \frac{36}{50}$. Hence the required length of the perpendicular is $PQ = \frac{\sqrt{457}}{5}$.

## 7.6 Illustrative Examples

**Example 7.13** Show that the line $\frac{x-1}{-2} = \frac{y+2}{3} = \frac{z+5}{4}$ lies on the plane $x + 2y - z = 0$.

**Solution.** Let $L: \dfrac{x+1}{-2} = \dfrac{y+2}{3} = \dfrac{z+5}{4} = t(say)$. If the coordinates of every point on the line $L$ lies on the on the plane $x+2y-z = 0$, then we say that the line $L$ lies on the plane. Coordinates of any point on the line $L$ are $(-1-2t, -2+3t, -5+4t)$. As $(-1-2t)+2(-2+3t)-(-5+4t) = 0$, the coordinates of any point on the line $L$ satisfy the equation of the plane $x + 2y - z = 0$. Hence the line $L$ lies on the plane $x + 2y - z = 0$.

**Example 7.14** Prove that the planes $2x-3y-7z = 0, 3x-14y-13z = 0$ and $8x - 31y - 33z = 0$ pass through one line.

**Solution.** Let $\alpha_1: 2x - 3y - 7z = 0$, $\alpha_2: 3x - 14y - 13z = 0$ and $\alpha_3: 8x - 31y - 33z = 0$.
First we find the line of intersection of the planes $\alpha_1$ and $\alpha_2$, say $L$ in the symmetric form. Both the planes pass through the origin $O(0, 0, 0)$. Hence the origin lies on the line of intersection $L$. Let $a, b, c$ be $d.r.s.$ of the line $L$. $L$ is perpendicular to the normals of the planes $\alpha_1$ and $\alpha_2$. Hence, we get $2a-3b-7c = 0$ and $3a-4b-13c = 0$. Solving these two equations for $a, b$ and $c$, we have $\frac{a}{-59} = \frac{b}{5} = \frac{c}{-19}$. Therefore $d.r.s.$ of the line are $59, -5, 19$. Equations of the line are $\frac{x-0}{59} = \frac{y-0}{-5} = \frac{z-0}{19} = t(say)$. Now we show that



the line $L$ lies on the plane $8x - 31y - 33z = 0$. Coordinates of any point on the line $L$ are$(59t, -5t, 19t)$. Now $8(59t) - 31(-5t) - 33(19t) = 0$. Therefore the coordinates of any point on the line $L$ satisfy the equation of the plane $8x - 31y - 33z = 0$. Hence the planes $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$ and $8x - 31y - 33z = 0$ pass through one line.

**Example 7.15** Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar.

**Solution.** Let $L_1: \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $L_2: \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$.
The points $(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (2, 3, 4)$ lie on the lines $L_1$ and $L_2$ respectively. The $d.r.s.$ of the line $L_1$ are $2, 3, 4$; and that of $L_2$ are $3, 4, 5$.

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0.$$

Hence the given lines $L_1$ and $L_2$ are coplanar.

**Example 7.16** Find the length and equations of the shortest distance between the lines

$$3x - 9y + 5z = \quad 0 \quad = x + y - z; \tag{9}$$
$$6x + 8y + 3z - 13 = \quad 0 \quad = x + 2y + z - 3. \tag{10}$$

**Solution.** Let $L_1: 3x-9y+5z = 0 = x+y-z$ and $L_2: 6x+8y+3z-13 = 0 = x + 2y + z - 3$. The line $L_1$ passes through the origin.Let $a, b, c$ be $d.r.s.$ of the line $L_1$, which is perpendicular to the normals of the planes $3x - 9y + 5z = 0$ and $x + y - z = 0$. Hence,we get $3a - 9b + 5c = 0$ and $a+b-c = 0$. Solving these two equations for $a, b, c$, we have $\frac{a}{4} = \frac{b}{8} = \frac{c}{12}$. Therefore $d.r.s.$ of $L_1$ are $1, 2, 3$.
Now to find the coordinates of a point on the line $L_2$, take $z = 0$. We have $6x + 8y = 13$ and $x + 2y = 3$. Solving these two equations for $x$ and $y$, we get $x = \frac{1}{2}$, $y = \frac{5}{4}$. Therefore $A(\frac{1}{2}, \frac{5}{4}, 0)$ is a point on the line $L_2$. Let $e, f, g$ be $d.r.s.$ of the line $L_2$, which is perpendicular to the normals of the planes $6x + 8y + 3z = 13$ and $x + 2y + z = 3$. Using this we get

$6e + 8f + 3g = 13$ and $e + 2f + g = 3$. Solving these two equations for $e, f, g$, we have $\frac{e}{2} = \frac{f}{-3} = \frac{g}{4}$. Therefore $d.r.s.$ of the line $L_2$ are $2, -3, 4$.

Now let $L$ be the line of shortest distance between $L_1$ and $L_2$. Therefore $L$ is perpendicular to both $L_1$ and $L_2$. From this, we have $l + 2m + 3n = 0$ and $2l - 3m + 4n = 0$. Solving these two equations, we get $d.r.s$ of the line $L$ as $17, 2, -7$. Hence the $d.c.s.$ of the line $L$ are $\frac{17}{\sqrt{342}}, \frac{2}{\sqrt{342}}, \frac{-7}{\sqrt{342}}$ The length of the shortest distance between the lines $L_1$ and $L_2$ is the projection of the line segment $OA$ on the line $L$ equals

$$(\frac{1}{2} - 0)\frac{17}{\sqrt{342}} + (\frac{5}{4} - 0)\frac{2}{\sqrt{342}} + (0 - 0)\frac{-7}{\sqrt{342}} = \frac{11}{\sqrt{342}}.$$

The equation of the plane containing the lines $L$ and $L_1$ is

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 1 & 2 & 3 \\ 17 & 2 & -7 \end{vmatrix} = 0 \text{ i.e., } 10x - 29y + 16z = 0.$$

The equation of the plane containing the lines $L$ and $L_2$ is

$$\begin{vmatrix} x-\frac{1}{2} & y-\frac{5}{4} & z-0 \\ 2 & -3 & 4 \\ 17 & 2 & -7 \end{vmatrix} = 0 \text{ i.e., } 13x + 82y + 55z = 109.$$

Hence the equations of the line of shortest distance are

$$10x - 29y + 16z = 0 = 13x + 82y + 55z - 109.$$

## 7.7 Exercise

1. Show that the lines $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ and $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ are coplanar.

2. Show that the lines $\frac{x-3}{1} = \frac{y-5}{2} = \frac{z-1}{-1}$ and $\frac{x-4}{2} = \frac{y-2}{-1} = \frac{z-4}{2}$ are coplanar and find the equation of the plane passing through the lines.



3. Show that the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is $\frac{1}{\sqrt{6}}$.

4. Find the length and equations of the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

5. Find the distance of $(-1, 2, 5)$ from the line through $(3, 4, 5)$ having $d.c.s.$ are proportional to $2, -3, 6$.

6. Find the distance of the point $(6, 6, -1)$ from the line $\frac{x-2}{1} = \frac{y-1}{2} = \frac{z+3}{-1}$. Also find the coordinates of its foot.

7. Show that the line $\frac{x-2}{2} = \frac{y+2}{-1} = \frac{z-3}{4}$ lies in the plane $2x + 2y - z + 3 = 0$.

8. Find the equation of the plane containing the line $\frac{x+2}{2} = \frac{y+3}{3} = \frac{z-4}{-2}$ and the point $(0, 6, 0)$.

9. Find the distance of $A(1, -2, 3)$ from the line $PQ$, through $P(2, -3, 5)$, which makes equal angles with the coordinate axes.

10. Find the equations of the line of shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$.

11. Show that the shortest distance between the $z - axis$ and the line $ax + by + cz + d = 0; a'x + b'y + c'z + d' = 0$ is $\frac{cd' - c'd}{\sqrt{(ac' - a'c)^2 + (bc' - b'c)^2}}$.

12. Show that the lines $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ and $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ are coplanar. Also find the equation of the plane passing through the lines.

13. Find the distance of the point $(-2, 2, -3)$ from the line $\dfrac{x-3}{1} = \dfrac{y+1}{2} = \dfrac{z-2}{-4}$. Also find the coordinates of its foot.

14. Find the equation of the plane containing the point $(0, 7, -7)$ and the line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$. Also show that the line $x = \dfrac{1}{3}(7-y) = \dfrac{1}{2}(z+7)$ lies in the same plane.

15. Obtain the coordinates of the points where the line of shortest distance between the lines $\dfrac{x-23}{-6} = \dfrac{y-19}{-4} = \dfrac{z-25}{3}$ and $\dfrac{x-12}{-9} = \dfrac{y-1}{4} = \dfrac{z-5}{2}$ meets them.

16. Prove that the lines $\dfrac{x-a+d}{\alpha-\delta} = \dfrac{y-a}{\alpha} = \dfrac{z-a-d}{\alpha+\delta}$ and $\dfrac{x-b+c}{\beta-\gamma} = \dfrac{y-b}{\beta} = \dfrac{z-b-c}{\beta+\gamma}$ are coplanar. Also find the equation in which they lie.

17. Find the length and the equations of the common perpendicular to the lines $\dfrac{x-3}{1} = \dfrac{y-4}{1} = \dfrac{z+1}{1}; \dfrac{x+6}{2} = \dfrac{y+5}{4} = \dfrac{z-1}{-1}$.

18. prove that the lines $x = ay + b = cz + d$ and $x = \alpha y + \beta = \gamma z + \delta$ are coplanar if $(\gamma - c)(a\beta - b\alpha) - (\alpha - a)(c\delta - d\gamma) = 0$.

19. Show that the equation to the plane containing the line $\dfrac{y}{b} + \dfrac{z}{c} = 1; x = 0$ and parallel to the line $\dfrac{x}{a} - \dfrac{z}{c} = 1; y = 0$ is $\dfrac{x}{a} - \dfrac{y}{b} - \dfrac{z}{c} = 1$. Also prove that $\dfrac{1}{d^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$, where $2d$ is the shortest distance between the lines.

20. Show that the lines $\dfrac{x+4}{3} = \dfrac{y+6}{5} = \dfrac{z-1}{-2}$ and $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ are coplanar.Also find the equation of the plane containing the lines.



21. Show that the lines $\dfrac{x}{\alpha} = \dfrac{y}{\beta} = \dfrac{z}{\gamma}, \dfrac{x}{a\alpha} = \dfrac{y}{b\beta} = \dfrac{z}{c\gamma}$ and $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ lie in a plane if $\dfrac{l}{\alpha}(b-c) + \dfrac{m}{\beta}(c-a) + \dfrac{n}{\gamma}(a-b)$.

## 7.8 Answers

(2) $3x - 4y - 5z + 16 = 0$ (4) $3\sqrt{30}, \frac{x-3}{2} \frac{y-8}{5} = \frac{z-3}{-1}$ (5) $\frac{4\sqrt{61}}{7}$
(6) $\sqrt{21}, (4, 5, -5)$ (8) $3x + 2y + 6z = 12$
(9) $\sqrt{\frac{14}{3}}$ (10) $11x + 2y - 7z + 6 = 0; 7x + y - 5z + 7 = 0$
(12) $11x - 6y - 5z = 67$ (13) $\sqrt{28}, (4, 1, -2)$
(14) $x + y + z = 0$ (15) $(11, 11, 31)$ and $(3, 5, 7)$ (16) $x - 2y + z = 0$
(17) $\sqrt{14}, \frac{x-1}{3} = \frac{y-2}{-1} \frac{z-1}{2}$ (20) $45x - 17y + 25z + 53 = 0$.

# Chapter 8

# The Sphere

**Definition 8.1** A sphere is the locus of a point which remains at a constant distance from a fixed point.
The constant distance is called the radius and the fixed point is called the centre of the sphere.

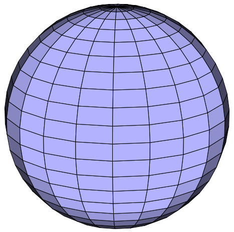


Figure 8.1: Sphere

## 8.1 Equation of a sphere:

We shall discuss various forms of the equation of sphere.

1. Standard equation of sphere: To find the equation of a sphere with centre at origin $0(0,0,0)$ and radius $r$.
   Let $P(x,y,z)$ be any point on the sphere. Then $OP = r$, therefore $OP^2 = r^2$. By distance formula $OP^2 = x^2 + y^2 + z^2$
   $\therefore x^2 + y^2 + z^2 = r^2$.
   This is standard equation of a sphere.

2. Centre Radius form of sphere: To find the equation of a sphere with centre at $C(a,b,c)$ and radius $r$.





Let $P(x,y,z)$ be any point on the sphere. Then $CP = r$, therefore $CP^2 = r^2$. By distance formula,

$$CP^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

$\therefore (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \qquad (1)$
This is centre radius form of sphere.

**Remark 8.1** 1. The equation (1) can be written in the form
$x^2 - 2ax + a^2 + y^2 - 2by + b^2 + z^2 - 2cz + c^2 = r^2$
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (2)$
Where $u = -a,\ v = -b,\ w = -c,\ d = a^2 + b^2 + c^2 - r^2$.
The equation (2) is called as general equation of sphere.

2. Note the following characteristics of the equation (2) of the sphere:

   (i) It is second degree equation in $x, y, z$;

   (ii) The coefficient of $x^2, y^2, z^2$ are all equal;

   (iii) The product terms $xy, yz, zx$ are absent.

Conversely, we shall now show that the general equation
$ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0,\ \ a \neq 0 \qquad (3)$
having the above three characteristics represents a sphere.
Dividing equation (3) by $a$, we get

$$x^2 + y^2 + z^2 + 2\frac{u}{a}x + 2\frac{v}{a}y + 2\frac{w}{a}z + \frac{d}{a} = 0.$$

Completing the square, we get

$$\left(x + \frac{u}{a}\right)^2 + \left(y + \frac{v}{a}\right)^2 + \left(z + \frac{w}{a}\right)^2 = \frac{u^2 + v^2 + w^2 - ad}{a^2}$$

Thus equation (3) represents a sphere with centre at $\left(\frac{-u}{a}, \frac{-v}{a}, \frac{-w}{a}\right)$ and radius $\frac{1}{|a|}\sqrt{u^2 + v^2 + w^2 - ad}$

**Example 8.1** Find the centre and radius of sphere

$$x^2 + y^2 + z^2 - 2cx - 2cy - 2cz + 2c^2 = 0.$$

**Solution:** Comparing given equation of sphere with general equation of sphere we have $u = -c, v = -c, w = -c, d = 2c^2$. Hence, centre is $C(-u, -v, -w) = (c, c, c)$ and radius is

$$\sqrt{u^2 + v^2 + w^2 - d} = \sqrt{c^2 + c^2 + c^2 - 2c^2} = |c|.$$

**Example 8.2** Find the centre and radius of the sphere
(i) $x^2 + y^2 + z^2 - 2x + 4y + 6z + 5 = 0$
(ii) $3x^2 + 3y^2 + 3z^2 + 6x - 9y - 12z + 15 = 0.$

**Solution:** (i) The given equation of the sphere is
$x^2 + y^2 + z^2 - 2x + 4y + 6z + 5 = 0.$
Comparing this equation with general equation or sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

We have $u = -1, v = 2, w = 3, d = 5.$
Centre of the sphere $= C(-u, -v, -w) = (1, -2, -3)$
Radius $= \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{1 + 4 + 9 - 5} = \sqrt{9} = 3.$
(ii) The equation of the given sphere is
$3x^2 + 3y^2 + 3z^2 + 6x - 9y - 12z + 15 = 0.$
Dividing by 3, we get $x^2 + y^2 + z^2 + 2x - 3y - 4z + 5 = 0.$
Comparing this equation of sphere with general equation of sphere.

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

We have $u = +1, v = \frac{-3}{2}, w = -2, d = 5$. Hence, the centre of the sphere is $C(-u, -v, -w) = (-1, \frac{3}{2}, 2)$ and

$$\text{radius} \;=\; \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{1 + \frac{9}{4} + 4 - 5} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$$



**Example 8.3** Find the equation of a sphere having centre $(2, -3, 4)$ and radius 5.

**Solution:** Let $P(x, y, z)$ be a point on the sphere. Then distance of $P$ from centre = radius i.e.

$$\begin{aligned} (x-2)^2 + (y+3)^2 + (z-4)^2 &= 5^2 \\ x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 &= 0 \end{aligned}$$

This is required equation of sphere.

**Example 8.4** Find the equation of the sphere passing through the points $(3, 0, 2), (-1, 1, 1), (2, -5, 4)$ and having centre on the plane
$2x + 3y + 4z = 6.$

**Solution:** Let $(a, b, c)$ be the centre of the required equation of sphere. Therefore

$$(a-3)^2 + b^2 + (c-2)^2 = (a+1)^2 + (b-1)^2 + (c-1)^2.$$

Simplifying, we get $5a - 2b + 2c = 10$ (1)
Similarly we get $a + 5b - 2c = 16$ (2)
Since the centre $(a, b, c)$ lies on the given plane $2x + 3y + 4z = 6$ we have
$2a + 3b + 4c = 6$ (3)
Solving (1), (2) and (3) we get $a = 0, b = -2, c = 3.$

$$\text{Radius of sphere} \;=\; \sqrt{(3-0)^2 + (0+2)^2 + (2-3)^2} = \sqrt{14}.$$

Required equation of sphere is

$$x^2 + y^2 + z^2 + 4y - 6z - 1 \;=\; 0$$

**Example 8.5** Find the equation of the sphere passing through
$(1, 1, 2)$ and $(0, -2, 1)$ and its centre lies on the line

$$x - 1 = 2 - y = z + 1.$$

**Solution:** Let $C(a,b,c)$ be the centre of the sphere. Let $A(1,1,2)$ and $B(0,-2,1)$ be two points on the sphere. Thus, $AC^2 = BC^2 = (\text{radius})^2$ and we get

$$\begin{aligned}(a-1)^2+(b-1)^2+(c-2)^2 &= (a-0)^2+(b+2)^2+(c-1)^2\\ 2a+6b+2c &= 1 \qquad (1)\end{aligned}$$

Since centre $C(a,b,c)$ lies on the line $x-1=2-y=z+1$,

$a-1 \;=\; 2-b \;=\; c+1 \;=\; r$ say

$a=4+1,\ b=2-r,\ c=r-1$. Using these values in (1), we get $r=\frac{11}{2}$. Thus, $a=r+1=\frac{11}{2}+1=\frac{13}{2},\quad b=2-\frac{11}{2}=\frac{-7}{2},\quad c=\frac{11}{2}-1=\frac{9}{2}$. Therefore centre of sphere is $C(a,b,c)=\left(\frac{13}{2},\frac{-7}{2},\frac{9}{2}\right)$ and radius of sphere equals

$$\sqrt{\left(\frac{13}{2}\right)^2+\left(2-\frac{7}{2}\right)^2+\left(\frac{9}{2}-1\right)^2} = \sqrt{\frac{169+9+49}{4}}=\frac{\sqrt{227}}{2}$$

Required equation of sphere is

$$\begin{aligned}\left(x-\frac{13}{2}\right)^2+\left(y+\frac{7}{2}\right)^2+\left(z-\frac{9}{2}\right)^2 &= \frac{227}{2}\\ x^2+y^2+z^2-13x+7y-9z+18 &= 0.\end{aligned}$$

**Example 8.6** Find the equation of the sphere passing through $(0,3,0)$, $(2,1,-1)$ and whose centre lies on the line $x-y-z=0=2x+3y$.
**Solution:** Let $C(a,b,c)$ be the centre of the sphere. Since the center $C(a,b,c)$ lies on the line.

$$\begin{aligned}x-y-z &= 0=2x+3y\\ a-b-c &= 0 \qquad (1)\\ 2a+3b &= 0 \qquad (2)\end{aligned}$$

Let $A(2,1,-1)$ and $B(0,3,0)$ be two points on the sphere. Therefore $(AC)^2=(BC)^2=(\text{radius})^2$

$$\begin{aligned}(a-2)^1+(b-1)^2+(c+1)^2 &= a^2+(b-3)^2+c^2\\ -4a+4b+2c &= 3 \qquad (3)\end{aligned}$$



Solving (1), (2) and (3) we get $a=-\dfrac{9}{10},\ b=\dfrac{3}{5},\ c=-\dfrac{15}{10}$

$$\text{Radius of sphere} = \sqrt{\left(\frac{-9}{10}\right)^2+\left(\frac{6}{10}-3\right)^2+\left(\frac{-15}{10}\right)^2}=\sqrt{\frac{882}{100}}$$

The required equation of sphere is

$$\begin{aligned}\left(x+\frac{9}{10}\right)^2+\left(y-\frac{6}{10}\right)^2+\left(z+\frac{15}{10}\right)^2 &= \frac{882}{100}\\ x^2+y^2+z^2+9x-6y+15z-27 &= 0.\end{aligned}$$

### 8.1.1 Sphere with a given diameter:

To find the equation of the sphere described on the line joining the points $A(x_1,y_1,z_1), B(x_2,y_2,z_2)$ as diameter.

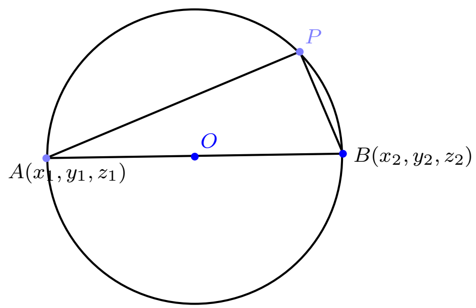



Figure 8.2: Diameter form of Sphere

Let $P(x,y,z)$ be any point on the sphere described on $AB$ as diameter. Since section of the sphere by the plane through the three points $P,A,B$ is a great circle having $AB$ as diameter and $P$ lies on semi-circle. Therefore $PA\perp PB$. The direction cosines of $PA$ and $PB$ are proportional to $x-x_1, y-y_1, z-z_1$ and $x-x_2, y-y_2, z-z_2$ respectively. Since $PA$ is perpendicular to $PB$,

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

This is diameter form of equation of the sphere.

**Example 8.7** Obtain the equation of the sphere described on the join of $A(2,-3,4), B(-5,6,-7)$ as diameter.

**Solution:** Note that $A, B$ are the end points of the diameter $AB$. The equation of sphere in diameter form is

$$\begin{aligned}(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2) &= 0\\(x-2)(x+5)+(y+3)(y-6)+(z-4)(z+7) &= 0\\x^2+y^2+z^2+3x-3y+3z-56 &= 0\end{aligned}$$

**Example 8.8** Obtain the equation of the smallest sphere passing through $A(-1,2,3)$ and $B(1,3,-4)$.

**Solution:** Let $P(x,y,z)$ be any point on the sphere. The direction ratios of $AP$ are $x+1, y-2, z-3$. The direction ratios of $BP$ are $x-1, y-3, z+4$. Since $AP \perp BP$

$$\begin{aligned}(x+1)(x-1)+(y-2)(y-3)+(2-3)(z+4) &= 0\\x^2+y^2+z^2-5y+z-7 &= 0.\end{aligned}$$

### 8.1.2 Intercept Form:

To find the equation of a sphere with intercepts $a, b, c$ on $x, y$ and $z$-axis respectively and passing through the origin. Let the equation of required sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad (1)$$

Since origin $O(0,0,0)$ lies on sphere (1) $d=0$. Since $x, y, z$ intercepts are $a, b, c$ respectively. Therefore the point $A(0,0,0)$ lies on sphere (1)

$$a^2+2ua=0 \text{ i. e. } u=-\frac{a}{2}$$



Similarly the points $B(0,b,0)$ and $C(0,0,c)$ lies on sphere (1). This gives $v=-\dfrac{b}{2}$ and $w=-\frac{c}{2}$. Using above values of $u, v, w, d$ in equation (1) we get

$$x^2+y^2+z^2-ax-by-cz=0$$

This is equation of sphere in intercept form.

**Example 8.9** Find the equation of the sphere passing through the origin and making equal positive intercept 5 units of the axes.

**Solution :** We know equation of the plane passing through origin and making intercept $a, b, c$ on co-ordinate axes.

$$x^2+y^2+z^2-ax-by-cz=0$$

In this case $a=b=c=5$. Hence, equation of the sphere is

$$x^2+y^2+z^2-5x-5y-5z=0$$

**Example 8.10** Find the equation of the sphere which circumscribes the tetrahedron: $(0,0,0),\ (0,3,0),\ (5,0,0),\ (0,0,7)$.

**Solution:** The equation of sphere passing through origin and making intercepts $a, b, c$ on co-ordinate axes is

$$\begin{aligned}x^2+y^2+z^2-ax-by-cz &= 0\\x^2+y^2+z^2-5x-3y-7z &= 0\end{aligned}$$

### 8.1.3 Equation of the sphere through four given points:

We want to find the equation of the sphere passing through four given points, $A(x_1,y_1,z_1),\ B(x_2,y_2,z_2),\ C(x_3,y_3,z_3)$ and $D(x_4,y_4,z_4)$.
Let $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ (1)
be the required equation of sphere.
Since the four points $A(x_1,y_1,z_1), B(x_2,y_2,z_2), C(x_3,y_3,z_3)$ and

$D(x_4, y_4, z_4)$ lie on the sphere (1), we get

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d &= 0 \qquad (ii) \\ x_2^2 + y_2^2 + z_2^2 + 2ux_2 + 2vy_2 + 2wz_2 + d &= 0 \qquad (iii) \\ x_3^2 + y_3^2 + z_3^2 + 2ux_3 + 2vy_3 + 2wz_3 + d &= 0 \qquad (iv) \\ x_4^2 + y_4^2 + z_4^2 + 2ux_4 + 2vy_4 + 2wz_4 + d &= 0 \qquad (v) \end{aligned}$$

Eliminating $u, v, w$ and $d$ from above five equations, we have

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

This is the required equation through the four given points.

**Note:** While solving the problems we may adopt a simple method as illustrated in solved examples.

**Example 8.11** Find the equation of the sphere passing through the points $A(2, 4, -1),\ B(0, -4, 3),\ C(-2, 0, 1)$ and $D(6, 0, 9)$.

**Solution :** Let $E(a, b, c)$ be the centre of the sphere. Since the points $A(2, 4, -1),\ B(0, -4, 3),\ C(-2, 0, 1)$ and $D(6, 0, 9)$ lies on the sphere. Therefore radius $= AE = BE = CE = DE$. This implies that $AE^2 = BE^2 = CE^2 = DE^2$. Now $AE^2 = BE^2$ gives

$$\begin{aligned} (a-2)^2 + (b-4)^2 + (c+1)^2 &= (a-0)^2 + (b+4)^2 + (6-3)^2 \\ a + 4b - 2c &= -1 \qquad (1) \end{aligned}$$

Similarly $AE^2 = CE^2$ gives

$$\begin{aligned} (a-2)^2 + (b-4)^2 + (c+1)^2 &= (a+2)^2 + b^2 + (c-1)^2 \\ 2a + 2b - c &= 4 \qquad (2) \end{aligned}$$

Next $AE^2 = DE^2$ gives

$$\begin{aligned} (a-2)^2 + (b-4)^2 + (c+1)^2 &= (a-6)^2 + b^2(c-9)^2 \\ 2a - 2b + 5c &= 24 \qquad (3) \end{aligned}$$



Adding (2) and (3)

$$4a + 4c = 28 \quad \text{i.e.} \quad a + c = 7 \qquad (4)$$

Multiplying equation (3) by 2 and adding to equation (1) we get

$$5a + 8c = 87 \qquad (5)$$

Solving (4) and (5) we obtain $a = 3,\ c = 4$ By (1) $b = 1$.
The centre of the sphere is $E(a, b, c) = (3, 1, 4)$ and its radius is $\sqrt{35}$. By centre radius form of sphere.

$$\begin{aligned} (x-a)^2 + (y-b)^2 + (z-c)^2 &= r^2 \\ \text{i.e. } (x-3)^2 + (y-1)^2 + (z-4)^2 &= 35 \\ x^2 + y^2 + z^2 - 6x - 2y - 8z - 9 &= 0 \end{aligned}$$

This is required equation of sphere.

**Example 8.12** Find the equation of the sphere passing through $O(0, 0, 0),\ A(0, 1, -1),\ B(-1, 2, 0)$ and $C(1, 3, 2)$.

**Solution:** Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (1)$
be the required equation of sphere. Since $O(0, 0, 0)$ lies on (1) , we get $d = 0$. Since the point $A(0, 1, -1)$ lies on (1)

$$0u + v - w + 1 = 0 \qquad (1)$$

Similarly the points $B(-1, 2, 0)$ and $C(1, 3, 2)$ lies on (1)

$$\begin{aligned} -2u + 4v + 0w + 5 &= 0 \qquad (3) \\ u + 3v + 2w + 7 &= 0 \qquad (4) \end{aligned}$$

Solving equations (2), (3) and (4) we get

$$u = \frac{-11}{14},\ \ v = \frac{-23}{14},\ \ w = \frac{-9}{14}.$$

Using these values in equation (1)

$$\begin{aligned} x^2 + y^2 + z^2 - \frac{11}{7}x - \frac{23}{7}y - \frac{9}{7}z &= 0 \\ 7x^2 + 7y^2 + 7z^2 - 11x - 23y - 9z &= 0 \end{aligned}$$

This is required equation of the sphere.

**Example 8.13** Find the equation of the sphere passing through $(1,1,0)$, $(0,-1,2)$, $(2,0,-1)$, $(2,2,0)$

**Solution :** Let $E(a,b,c)$ be the centre of the sphere. Since $A(1,1,0)$, $B(0,-1,2)$, $C(2,0,-1)$ and $D(2,2,0)$ lies on the sphere, radius $= AE = BE = CE = DE$. This implies that $(AE)^2 = (BE)^2 = (CE)^2 = (DE)^2$. Now $(AE)^2 = (BE)^2$ gives

$$\begin{aligned}(a-1)^2+(b-1)^2+c^2 &= a^2+(b+1)^2+(c-2)^2\\ 2a+4b-4c &= -3 \qquad (1)\end{aligned}$$

Similarly $(AE)^2 = (CE)^2$ gives

$$\begin{aligned}(a-1)^2+(b-1)^2+c^2 &= (a-2)^2+b^2+(c+1)^2\\ 2a-2b-2c &= 3 \qquad (2)\end{aligned}$$

$(AE)^2 = (DE)^2$ gives $a+b=3$ (4)
After solving (1), (2) and (3) we get $a = \frac{33}{10}$, $b = \frac{-3}{10}$, $c = \frac{21}{10}$.
The centre of the sphere is $C(a,b,c) = \left(\frac{33}{10}, \frac{-3}{10}, \frac{21}{10}\right)$ and its radius is $\frac{\sqrt{1139}}{10}$. By centre radius form of sphere

$$\begin{aligned}(x-a)^2+(y-b)^2+(z-c)^2 &= r^2\\ \left(x-\frac{33}{10}\right)^2+\left(y+\frac{3}{10}\right)^2+\left(z-\frac{21}{10}\right)^2 &= \frac{1139}{100}\end{aligned}$$

$5(x^2+y^2+z^2)-33x+3y-21z+20=0$. This is required equation of sphere.

**Example 8.14** Find the equation of the sphere passing through the points: $A(1,0,-1)$, $B(2,1,0)$, $C(1,1,-1)$ and $D(1,1,1)$.

**Solution:** Let $A(1,0,-1)$, $B(2,1,0)$, $C(1,1,-1)$ and $D(1,1,-1)$ be four points on the sphere. Let $E(a,b,c)$ be the centre of the sphere. Therefore radius $= AE = BE = CE = DE$. This implies $(AE)^2 = (BE)^2 = (CE)^2 = (DE)^2$. Now $(AE)^2 = (BE)^2$ gives

$$\begin{aligned}(a-1)^2+b^2+(c+1)^2 &= (a-2)^2+(b-1)^2+c^2\\ 2a+2b+2c &= 3 \qquad (1)\end{aligned}$$



Similarly $(AE)^2 = (CE)^2$ gives $2b+2c = 1$ (2)
$(CE)^2 = (DE)^2$ gives

$$\begin{aligned}(a-1)^2+(b-1)^2+(c+1)^2 &= (a-1)^2+(b-1)^2+(c-1)^2\\ c &= 0 \qquad (3)\end{aligned}$$

By (2) $b = \frac{1}{2}$ and by (1) $a = 1$. The centre of the sphere is $E(a,b,c) = (1, \frac{1}{2}, 0)$ and its radius is $\sqrt{0+\frac{1}{4}+1} = \frac{\sqrt{5}}{2}$. By centre radius form the sphere

$$\begin{aligned}(x-1)^2+(y-\frac{1}{2})^2+(z-0)^2 &= \frac{5}{4}\\ x^2+y^2+z^2-2x-y &= 0\end{aligned}$$

This is the required equation of sphere.

## 8.2 Plane section of a sphere:

To prove that the plane section of a sphere is a circle.

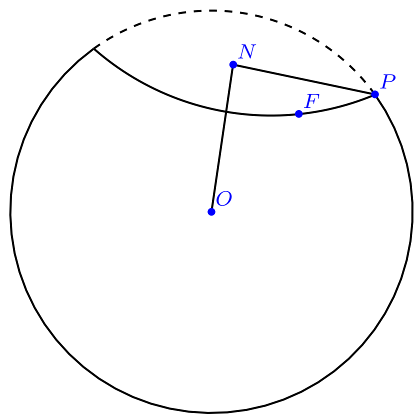



Figure 8.3:

Let $O$ be the centre of the sphere. Let $P$ be any point which is common to the sphere and the plane. Let $ON$ is perpendicular to the given plane; $N$ being the foot of the perpendicular. The line $NP$ lies in the plane. Therefore $ON \perp NP$. In right angled triangle $ONP$,
$OP^2 = ON^2 + NP^2$. Hence, $NP^2 = OP^2 - ON^2$
The points $O$ and $N$ being fixed. But $OP$ is radius of the sphere which is fixed. Also $ON$ is fixed. Therefore $OP^2 - ON^2$ is fixed $NP^2$ and hence $NP$ is constant. Hence the locus of $P$ is circle whose centre in $N$ (the foot of perpendicular from the centre of the sphere to the plane) and radius is $\sqrt{OP^2 - ON^2}$. Thus section of a sphere by a plane is a circle.

**Remark 8.2** 1. The section of a sphere by a plane through it centre is known as a great circle. The centre and radius of a great circle are the some as those of sphere.

2. (i) If $ON > OP$, then circle is imaginary.
(ii) If $ON = OP$, then it is the point circle.

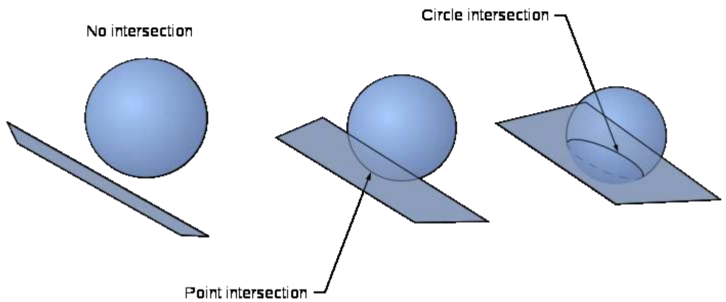


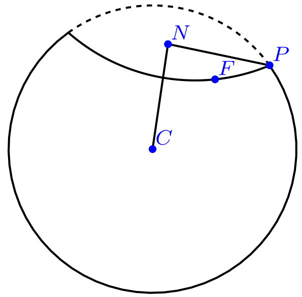

Figure 8.4: Section of a sphere by a plane

**Example 8.15** Find the centre and the radius of the circle
$x^2 + y^2 + z^2 - 2y - 4z = 11, \; x + 2y + 2z = 15.$

**Solution:** The given equation of circle is the intersection of the sphere
$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ (1)



with the plane $x + 2y + 2z - 15 = 0$ (2)
The centre of the sphere (1) is $C(-u, -v, -w) = (0, 1, 2)$, and radius $= r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{0 + 1 + 4 + 11} = \sqrt{16} = 4.$




Draw $CM$ perpendicular from $C$ on the plane $x + 2y + 2z - 15 = 0$ Since $CM$ is perpendicular to the plane (2), the direction ratios of $CM$ are $1, 2, 2$ and the equations of line $CM$ are $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{2} = r$ say
$x = r, y = 2r + 1, z = 2r + 2$ (3)
These are the co-ordinates of any point on the line $CM$. If this point lies on the plane (2) then

$$4 + 2(2r + 1) + 2(2r + 2) = 15 \text{ i. e. } r = 1$$

Putting $r = 1$ in (3) we get $M(x, y, z) = (1, 3, 4)$
From right angled triangle $CMP$, $CM^2 + MP^2 = CP^2$
$MP = \sqrt{CP^2 - CM^2} = \sqrt{16 - [1 + 4 + 4]} = \sqrt{7}.$
The centre of the circle is $M(1, 3, 4)$ and its radius $= \sqrt{7}.$

**Example 8.16** Prove that the straight line $\frac{x+1}{4} = \frac{y-2}{1} = \frac{z-2}{1}$ touches the sphere $x^2 + y^2 + z^2 = 9$. Find the point of contact.

**Solution** We have the equation of sphere
$x^2 + y^2 + z^2 = 9$ (1)
The equations of the line are $\frac{x+1}{4} = \frac{y-2}{1} = \frac{z-2}{1} = r$ say
$x = 4r - 1, y = r + 2, z = r + 2$ (2)
Let $P(x, y, z) = P(4r - 1, r + 2, r + 2)$ be points on the line.

If this point lies on sphere (1) then

$$\begin{aligned}(4r-1)^2+(r+2)^2+(r+2)^2 &= 9\\ 18r^2 &= 0\\ r &= 0\end{aligned}$$

Both roots of the quadratic equation in $r$ are zero. Putting $r=0$ in (3) we get $P(-1,2,2)$. Clearly $P(-1,2,2)$ satisfies equation of sphere (1). i.e. $P(-1,2,2)$ lies on (1). Therefore the straight line (2) is tangent line to the sphere (1) at point $P(-1,2,2)$. The point $P(-1,2,2)$ is required point of contact.

## 8.3 Intersection of two spheres:

To prove that the curves of intersection of two spheres is a circle.
Let

$$\begin{aligned}S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1 &= 0 \qquad (1)\\ S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2 &= 0 \qquad (2)\end{aligned}$$

be two equations of sphere. Consider the equation $S_1-S_2=0$ i.e.

$$2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+(d_1-d_2)=0 \qquad (3)$$

Equation (3) is first degree equation in $x,y,z$. Therefore it represents a plane. Let $P(x_1,y_1,z_1)$ be a point common to two spheres (1) and (2). Therefore

$$\begin{aligned}S_1 \equiv x_1^2+y_1^2+z_1^2+2u_1x_1+2v_1y_1+2w_1z_1+d_1 &= 0\\ S_2 \equiv x_1^2+y_1^2+z_1^2+2u_2x_1+2v_2y_1+2w_2z_1+d_2 &= 0\end{aligned}$$

Now at point $P(x_1,y_1,z_1), S_1-S_2=0$ gives

$$2(u_1-u_2)x_1+2(v_1-v_2)y_1+2(w_1-w_2)z_1+(d_1-d_2)=0$$

This shows that the point $P(x_1,y_1,z_1)$ satisfy equation of plane (3)
Therefore the curves of intersection of two spheres (1) and (2) is same as



that of sphere (1) and plane (3) or sphere (2) and plane (3).
But intersection of a sphere and a plane is a circle.
Thus the curves of intersection of two spheres is a circle.

**Note:** The plane given by equation (3) is called radical plane.

**Example 8.17** Find the equation of the sphere through the circle $x^2+y^2+z^2+6x-4y-6z-14=0, x+y-z=0$ and passing through the point $(1,1,-1)$. Also find centre and radius of this sphere.

**Solution:** The equations of given circle are

$$\begin{aligned}S \equiv x^2+y^2+z^2+6x-4y-6z-14 &= 0\\ U \equiv x+y-z &= 0\end{aligned}$$

The equation of the sphere passing through this circle is $S+\lambda U=0$, where $\lambda$ is a real number.

$$x^2+y^2+z^2+6x-4y-6z-14+\lambda(x+y+z)=0 \qquad (1)$$

Since the point $(1,1,-1)$ lies on the sphere (1)

$$1+1+1+6-4+6-14+\lambda(1+1+1) = 0.$$

Hence, $\lambda=1$. Putting $\lambda=1$ in equation (1) we get required equation of sphere

$$\begin{aligned}x^2+y^2+z^2+6x-4y-6z-14+(x+y-z) &= 0\\ x^2+y^2+z^2+7x-3y-7z-14 &= 0\end{aligned}$$

The centre of this sphere is $C\left(\frac{-7}{2},\frac{3}{2},\frac{7}{2}\right)$ and its radius is $\frac{\sqrt{163}}{2}$.

**Example 8.18** Show that following spheres touches each other and find their point of touching

$$\begin{aligned}x^2+y^2+z^2-4x-2y-4z+5 &= 0 \qquad (1)\\ x^2+y^2+z^2-6x-6y+17 &= 0 \qquad (2)\end{aligned}$$

**Solution:** The centre of sphere (1) is $C_1(2,1,2)$ and radius is $r_1 = \sqrt{4+1+4-5} = \sqrt{4} = 2$. The centre of sphere (2) is $C_2(3,3,0)$ and radius is $r_2 = \sqrt{9+9+0-17} = \sqrt{1} = 1$.
The distance between the centre of the spheres
$= C_1\,C_2 = \sqrt{1+4+4} = \sqrt{9} = 3 = r_1 + r_2$.
Therefore two spheres (1) and (2) touch externally.
If $P(x,y,z)$ is their point of contact, then $P$ divides $C_1C_2$ internally in the ratio $r_1 : r_2$ i.e. $2 : 1$ therefore

$$x = \frac{2(3)+1(2)}{2+1} = \frac{8}{3}, y = \frac{2(3)+1(1)}{2+1} = \frac{7}{3}, z = \frac{2(0)+1(2)}{2+1} = \frac{2}{3}.$$

$P\left(\frac{8}{3}, \frac{7}{3}, \frac{2}{3}\right)$ is required point of contact.

### 8.3.1 Equations of a circle:

We know that the plane section of a sphere is a circle. Therefore the circle can be represented by two equations, one being of a sphere and other of the plane. Thus the two equations

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \quad lx+my+nz=P$$

taken together represent a circle. A circle can also be represented by the equations of any two spheres through it. We note that the equations $x^2 + y^2 + 2gx + 2fy + c = 0, \quad z = 0$ also represents a circle which is the intersection of the cylinder $x^2+y^2+2gx+2fy+c=0$ with the plane.

## 8.4 Sphere through a given Circle:

### 8.4.1 Sphere passing through the circle of intersection of the given sphere and plane:

To find the equation of a sphere which passes through the circle with equations

$$\begin{aligned} S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d &= 0 \qquad (1)\\ U \equiv lx+my+nz-p &= 0 \qquad (2) \end{aligned}$$



Consider the equation $S + \lambda U = 0$ i.e.

$$x^2+y^2+z^2+2ux+2vy+2wz+d+\lambda(lx+my+nz-p)=0$$

$$x^2+y^2+z^2+(2u+\lambda l)x+(2v+\lambda m)y+(2w+n\lambda)z+d-\lambda p=0(3)$$

In equation (3) we observe that it is the second degree equation in $x, y, z$ in which coefficients of $x^2, y^2, z^2$ are same and the product terms in $xy, yz, zx$ are absents. Therefore equation (3) represents a sphere. Let $(x_1, y_1, z_1)$ be any point which satisfy the equations of circle (1) and (2) i.e.

$$\begin{aligned} x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d &= 0\\ lx_1+my_1+nz_1-p &= 0 \end{aligned}$$

The point $(x_1, y_1, z_1)$ also satisfies equation (3)
$x_1^2+y_1^2+z_1^2+(2u+\lambda l)x_1+(2v+\lambda m)y_1+(2w+n\lambda)z_1+d-\lambda p$
$=(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)+\lambda(lx_1+my_1+nz_1-p)$
$=0+\lambda 0=0$
i.e. the point $(x_1, y_1, z_1)$ which is common to (1) and (2) also lies on (3). Hence (3) represents a sphere which passes through the circle in which the sphere (1) is cut by the plane (2). For different values of $\lambda$, we get different spheres. Thus, the equation $S + \lambda U = 0$ represents a sphere containing the circle $S = 0$ and $U = 0$ for all real values of $\lambda$.

### 8.4.2 Sphere passing through a circle, which is the intersection of two spheres:

Let $S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \qquad (1)$
and $S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \qquad (2)$
be two intersecting spheres. Equations (1) and (2) represents a circle. Consider the equation $S_1 + \lambda S_2 = 0 \quad (\lambda \neq -1)$
$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1+\lambda$
$(x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2)=0$
$(1+\lambda)x^2+(1+\lambda)y^2+(1+\lambda)z^2+2(u_1+\lambda u_2)x+2(v_1+\lambda v_2)y$
$+2(w_1+\lambda w_2)z+(d_1+\lambda d_2)=0 \qquad (3)$

Thus, equation (3) is a second degree equation in $x, y, z$ in which coefficients of $x^2, y^2, z^2$ are equal and product terms $xy, yz, zx$ are absent. Therefore equation (3) represents sphere.
If $P(x_1, y_1, z_1)$ is a point common to sphere (1) and (2) then

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d_1 &= 0 \\ x_1^2 + y_1^2 + z_1^2 + 2u_2x_1 + 2v_2y_1 + 2w_2z_1 + d_2 &= 0 \end{aligned}$$

At point $P(x_1, y_1, z_1)$

$$\begin{aligned} S_1 + \lambda S_2 &= (x_1^2 + y_1^2 + z_1^2 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d_1) \\ &\quad + \lambda(x_1^2 + y_1^2 + z_1^2 + 2u_2x_1 + 2v_2y_1 + 2w_2z_1 + d_2) \\ &= 0 + \lambda(0) = 0 \end{aligned}$$

Therefore $P(x_1, y_1, z_1)$ satisfies equation (3).
Thus $S_1 + \lambda S_2 = 0 \ (\lambda \neq -1)$ is a sphere through the circle of intersection of the two spheres $S_1 = 0, S_2 = 0$.

## 8.5 Intersection of a sphere and a line:

Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (1)
be the equation of the sphere. The equations of line having $l, m, n$ asdirection cosines and passing through $P(x_1, y_1, z_1)$ are $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ say. (2)
Therefore $x = x_1 + lr, \ \ y = y_1 + mr, \ \ z = z_1 + nr$ (3)
These are the co-ordinates of general point on the line (2).
If the point $(x, y, z) = (x_1 + lr, y_1 + mr, z_1 + nr)$ on the sphere (1), then
$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 + 2u(x_1 + lr) + 2v(y_1 + mr)$
$+ 2w(z_1 + nr) + d = 0$
$r^2(l^2 + m^2 + n^2) + 2r(lx_1 + my_1 + nz_1 + lu + mv + nw)$
$+ x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$
This is quadratic equation in $r$ and hence it gives two values of $r$ say $r_1$ and $r_2$. Therefore, there are two points $A(x_1 + lr_1, y_1 + mr_1, z_1 + nr_1)$ and $B(x_1 + lr_2, y_1 + mr_2, z_1 + nr_2)$ common to sphere (1) and line (2). Hence in general a line intersects a sphere in two points.



**Remark 8.3**
1. If $r_1$ and $r_2$ are real and distinct, then there are two common points.
2. If $r_1 = r_2$, then line touches the sphere.
3. If $r_1$ and $r_2$ are imaginary, then the line does not intersect the sphere.

**Example 8.19** Find the point at which the line $\frac{x-7}{2} = \frac{y-6}{1} = \frac{z+5}{-1}$ Cuts the sphere $x^2 + y^2 + z^2 - 2x + 3y - 5z - 31 = 0$.

**Solution:** We have equation of sphere

$$x^2 + y^2 + z^2 - 2x + 3y - 5z - 31 = 0. \qquad (1)$$

The equations of the line are

$$\frac{x-7}{2} = \frac{y-6}{1} = \frac{z+5}{-1} = r \text{ say} \qquad (2)$$

Therefore $x = 2r + 7, y = r + 6, z = -r - 5$.
Let $P(x, y, z) = (2r + 7, r + 6, -r - 5)$ be a point on the line. If this point lies on sphere (1) then
$(2r+7)^2 + (r+6)^2 + (-r-5)^2 - 2(2r+7) + 3(r+6) - 5(-r-5) - 31 = 0.$
Hence,

$$\begin{aligned} 6r^2 + 54r + 108 &= 0 \\ (r + 6)(r + 3) &= 0 \\ r = -6 \text{ or } &\quad r = -3 \end{aligned}$$

Putting $r = -6$ and $r = -3$ in the coordinates of $P$, we obtain point of intersection of line and sphere as $P(-5, 0, 1), \ Q(1, 3, -2)$.

**Example 8.20** Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 + 2x - 2y - 2z - 1 = 0; 2x - 2y + z - 1 = 0$ and passing through the point $(3, -1, 1)$.

**Solution:** The equations of given circle are

$$\begin{aligned} S \equiv x^2 + y^2 + z^2 + 2x - 2y - 2z - 1 &= 0 \\ U \equiv 2x - 2y + z - 1 &= 0. \end{aligned}$$

The equation of the sphere passing through this circle is $S + \lambda U = 0$, where $\lambda$ is real number.

$$x^2 + y^2 + z^2 + 2x - 2y - 2z - 1 + \lambda(2x - 2y + z - 1) = 0 \qquad (1)$$

Since the point $(3, -1, 1)$ lies on (1)

$$8\lambda + 16 = 0 \text{ i. e. } \lambda = -2.$$

Putting $\lambda = -2$ in equation (1) we get.

$$\begin{aligned} x^2 + y^2 + z^2 + 2x - 2y - 2z - 1 - 2(2x - 2y + z - 1) &= 0 \\ x^2 + y^2 + z^2 - 2x + 2y - 4z + 1 &= 0 \end{aligned}$$

## 8.6 Equation of Tangent Plane:

### 8.6.1 Standard equation of sphere:

To find the equation of a tangent plane to the standard equation of sphere $x^2 + y^2 + z^2 = a^2$ at $P(x_1, y_1, z_1)$ on it.
Let $x^2 + y^2 + z^2 = a^2$ (1)
be the standard equation of the sphere.
Let $P(x_1, y_1, z_1)$ be any point on the sphere (1)
$\therefore x_1^2 + y_1^2 + z_1^2 = a^2$ (2)
The equations of a line passing through $P(x_1, y_1, z_1)$ are
$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ say (3)
The co-ordinates of a general point on the line are

$$x = x_1 + lr, \;\; y = y_1 + mr, \;\; z = z_1 + nr.$$

This point lies on the sphere (1) if

$$\begin{aligned} (x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 &= a^2 \\ r^2(l^2 + m^2 + n^2) + 2r(lx_1 + my_1 + nz_1) + (x_1^2 + y_1^2 + z_1^2) &= a^2 \end{aligned}$$



By using equation (2)

$$\begin{aligned} r^2(l^2 + m^2 + n^2) + 2r(lx_1 + my_1 + nz_1) + a^2 &= a^2 \\ r^2(l^2 + m^2 + n^2) + 2r(lx_1 + my_1 + nz_1) &= 0 \end{aligned}$$

One root of this equation is $0$. Therefore, in order that the line (3) is tangent line to sphere (1), the other root must also be $0$.
Therefore $lx_1 + my_1 + nz_1 = 0$ (4)
The equation of tangent plane at $P(x_1, y_1, z_1)$ is obtained by eliminating $l, m, n$ from equations (3) and (4)

$$\begin{aligned} (x - x_1)x_1 + (y - y_1)y_1 + (z - z_1)z_1 &= 0 \\ xx_1 + yy_1 + zz_1 &= x_1^2 + y_1^2 + z_1^2 \\ xx_1 + yy_1 + zz_1 &= a^2 \end{aligned}$$

This is the equation of tangent plane to sphere (1) at $P(x_1, y_1, z_1)$ on it.

### 8.6.2 Equation of Tangent Plane:

To find the equation of the tangent plane at any point $(x_1, y_1, z_1)$ of the sphere.
Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (1)
be the general equation of the sphere. Since the point $P(x_1, y_1, z_1)$ lies on the sphere (1)
$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$ (2)
The equation of a line passing through $P(x_1, y_1, z_1)$ are
$\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n} = r$ say (3)
Then the co-ordinates of general point on this line are
$x = x_1 + lr, \; y = y_1 + mr, \; z = z_1 + nr.$
This point lies on the sphere (1) if
$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 + 2u(x_1 + lr) + 2v(y_1 + mr)$
$+ 2w(z_1 + nr) + d = 0$
$r^2(l^2 + m^2 + n^2) + 2r[l(x_1 + u) + m(y_1 + v) + n(z_1 + w)]$
$(x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$
By using equation (2)

$r^2(l^2+m^2+n^2)+2r[l(x_1+u)+m(y_1+v)+n(z_1+w)]=0$
One root of this equation is $0$.
Therefore, in order that the line (3) is a tangent line to sphere (1), the other root must also be $0$.
$l(x_1+u)+m(y_1+v)+n(z_1+w)=0 \qquad (4)$
The equation of tangent plane at $P(x_1,y_1,z_1)$. is obtained by eliminating $l,m,n$ from (3) and (4).
$(x-x_1)(x_1+u)+(y-y_1)(y_1+v)+(z-z_1)(z_1+w)=0$
$xx_1+yy_1+zz_1+ux+vy+wz=x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1$
Adding $ux_1+vy_1+wz_1+d$ on both the sides
$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$
$=x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d.$
$\therefore$ By using (2)
$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0.$
This is the equation of the tangent plane at $P(x_1,y_1,z_1)$ to the sphere (1).

**Example 8.21** Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$. Also find the point of contact.

**Solution:** We have equation of sphere

$$x^2+y^2+z^2-2x-4y+2z-3=0 \qquad (1)$$

The centre of sphere (1) is $(1,2,-1)$ and radius is
$r=\sqrt{u^2+v^2+w^2-d}=\sqrt{1+4+1+3}=\sqrt{9}=3.$
The length of perpendicular from the centre. $C(1,2,-1)$ to the plane $2x-2y+z+12=0$ is

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|=\left|\frac{2(1)-2(2)+1(-1)+12}{\sqrt{4+4+1}}\right|=\left|\frac{9}{3}\right|=3,$$

which is radius of the sphere. Therefore the plane $2x-2y+z+12=0$ touches the sphere (1). The direction ratios of normal to the plane (2) are $2,-2,1$.



$CM$ is normal to the plane (2) Equation of line $CM$ are
$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}=r$ say. Hence,
$x=2r+1, y=-2r+2, z=r-1 \qquad (3)$
These are the co-ordinates of any point on the line. If this point lies on the plane (2)

$$\begin{aligned}2(2r+1)-2(-2r+2)+(r-1)+12 &= 0\\ r &= -1.\end{aligned}$$

Put $r=-1$ in (3) we get point of contact $M(-1,4,-2)$.

### 8.6.3 The condition of tangency:

**Standard Sphere:**

To find the condition that the plane

$$lx+my+nz=p \qquad (1)$$

is tangent plane to the sphere $x^2+y^2+z^2=a^2 \qquad (2)$
Suppose the plane (1) is a tangent plane to the sphere (2) at $P(x_1,y_1,z_1)$ on it. We know that the equation of the tangent plane at $P(x_1,y_1,z_1)$ to sphere (2) is

$$xx_1+yy_1+zz_1=a^2 \qquad (3)$$

Since the equations (1) and (3) represents the same plane. Therefore, their coefficients are proportional i.e.

$$\frac{x_1}{l}=\frac{y_1}{m}=\frac{z_1}{n}=\frac{a^2}{p}$$

$$x_1=\frac{a^2l}{p}, y_1=\frac{a^2m}{p}, z_1=\frac{a^2n}{p}$$

Since the point $P(x_1,y_1,z_1)=P\left(\frac{a^2l}{p},\frac{a^2m}{p},\frac{a^2n}{p}\right)$ lies on sphere (2)

$$\begin{aligned}\left(\frac{a^2l}{p}\right)^2+\left(\frac{a^2m}{p}\right)^2+\left(\frac{a^2n}{p}\right)^2 &= a^2\\ a^2(l^2+m^2+n^2) &= p^2\end{aligned}$$

i.e. $p = \pm a\sqrt{l^2+m^2+n^2} = \pm a$.
This is required condition. If this condition is satisfied, then the point of contact is

$$(x_1, y_1, z_1) \quad = \quad \left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right) = (\pm al, \, \pm am, \, \pm an)\,.$$

**General equation of sphere:**

To find the condition that the plane $lx+my+nz=p$ is tangent plane to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad (1)$$

(1) is the general equation of the sphere having centre $C(-u,-v,-w)$ and radius $\sqrt{u^2+v^2+w^2-d}$.
The plane $lx+my+nz=p$ will be tangent plane to the sphere (1) if the length of perpendicular from the centre of the sphere is equal to the radius of the sphere.

$$\left|\frac{l(-u)+m(-v)+n(-w)-p}{\sqrt{l^2+m^2+n^2}}\right| \quad = \quad \sqrt{u^2+v^2+w^2-d}$$

Thus,$(lu+mv+nw+p)^2 = (u^2+v^2+w^2-d)(l^2+m^2+n^2)$ i.e.

$$(lu+mv+nw+p)^2 = u^2+v^2+w^2-d.$$

This is the required condition.

**Remark 8.4** Let $C$ and $r$ be the centre and radius of the sphere respectively and $P$ be any point in space. If
(i) $CP < r$, then the point $P$ lies inside the sphere.
(ii) $CP = r$, then the point $P$ lies on the sphere.
(iii) $CP > r$, then the point $P$ lies out side the sphere.



## 8.7 llustrative Examples:

**Example 8.22** Find the co-ordinates of the points where the line $\dfrac{x+3}{4} = \dfrac{y+4}{3} = \dfrac{z-8}{-5}$ intersects the sphere $x^2+y^2+z^2+2x-10y-23=0$.

**Solution:** The given equations of line are

$$\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5} \quad = \quad r \text{ say} \qquad (1)$$

Hence, $x = 4r-3, y = 3r-4, z = -5r+8$ (2)
Since line (1) cuts the equation of sphere.

$$x^2+y^2+z^2+2x-10y-23=0 \qquad (3)$$

$\therefore$ Using (2) in (3)
$(4r-3)^2+(3r-4)^2+(-5r+8)^2+2(4r-3)-10(3r-4)-23=0$

$$\begin{aligned} 50r^2-150r+100 &= 0 \\ (r-1)(r-2) &= 0 \\ r &= 1, r=2. \end{aligned}$$

When $r=1$, by (2) $P(x,y,z) = (1,-1,3)$ and when $r=2$, $Q(x,y,z) = (5,2,-2)$. Thus, $P(1,-1,3)$ and $Q(5,2,-2)$ are the required points of intersection.

**Example 8.23** Find the equation of the circle which is a section of the sphere $x^2+y^2+z^2+6y-6z-21=0$ and has its centre at the point $M(2,-1,2)$.

**Solution:** The given equation of the sphere is

$$x^2+y^2+z^2+6y-6z-21=0 \qquad (1)$$

The centre of the sphere (1) $= C(0,-3,3)$ and radius of sphere is $r = \sqrt{0+9+9+21} = \sqrt{39}$. The direction ratios of line $CM$ are $2,2,-1$. But $CM$ is normal to the plane.

Therefore equation of plane passing through $M(2,-1,2)$ and having direction ratios $2,2,-1$ of its normal is

$$\begin{aligned} a(x-x_1)+b(y-y_1)+c(z-z_1) &= 0 \\ 2(x-2)+2(y+1)-(z-2) &= 0 \\ 2x+2y-z &= 0 \end{aligned}$$

Therefore required equation of circle is
$x^2+y^2+z^2+6y-6z-21=0,\ \ 2x+2y-z=0.$

**Example 8.24** Find the equation of the sphere passing through the circle $x^2+y^2+z^2+2x+3y-6=0, x-2y+4z=9=0$ and through the center of the sphere $x^2+y^2-z^2-2x+4y-6z+5=0$.

**Solution:** Let

$$\begin{aligned} S \equiv x^2+y^2+z^2+2x+3y-6 &= 0 \\ U \equiv x-2y+4z-9 &= 0 \end{aligned}$$

We know $S+\lambda U=0$ represents sphere.

$$\therefore (x^2+y^2+z^2+2x+3y-6)+\lambda(x-2y+4z-9)=0 \qquad (1)$$

The centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$ is $(1,-2,3)$
Since sphere (1) passes through centre $(1,-2,3)$
Therefore $4+8\lambda=0,\ \lambda=\frac{-1}{2}$ Putting $\lambda=\frac{-1}{2}$ in equation (1) we get

$$2(x^2+y^2+z^2)+3x+8y-4z-3=0$$

This is required equation of sphere.

**Example 8.25** Find the equation of the sphere whose centre is $(-2,0,1)$ and which touches the plane $5x-y+4z=36$.

**Solution:** Radius of sphere equals the length of perpendicular drawn from $(-2,0,1)$ on the plane $5x-y+4z-36=0$. Thus,

$$\left|\frac{-10-0+4-36}{\sqrt{25+1+16}}\right| = \left|\frac{-42}{\sqrt{42}}\right| = \sqrt{42}$$



Since centre of the sphere is $(-2,0,1)$. By centre radius form of sphere

$$\begin{aligned} (x+2)^2+(y-0)^2+(z-1)^2 &= 42 \\ x^2+y^2+z^2+4x-2z-37 &= 0 \end{aligned}$$

**Example 8.26** Find the equation of the sphere passing through the points $P(2,3,-1)$ and $Q(1,1,0)$ and whose centre lies on the line

$$\frac{x}{3}=\frac{y+1}{-2}=\frac{z-2}{2}.$$

**Solution:** The given equations of line are $\frac{x}{3}=\frac{y+1}{-2}=\frac{z-2}{2}=r$ say

$$x=3r, y=-2r-1, z=2r+2.$$

Let $C(3r,-2r-1,2r+2)$ be the centre of the sphere. But $P(2,3,-1)$ and $Q(1,1,0)$ lies on sphere. Therefore $(\text{radius})^2=CP^2=CQ^2$.
$(3r-2)^2+(1+2r+3)^2+(2+2r+1)^2$

$$\begin{aligned} &= (3r-1)^2+(1+2r+1)^2+(2r+2-0)^2 \\ 6r &= -20 \ \text{ i.e. } r=\frac{-10}{3}. \end{aligned}$$

Centre of sphere $=C\left(-10,\frac{17}{3},\frac{-14}{3}\right)$ and radius of sphere is $\frac{\sqrt{1481}}{3}$.
By centre radius form required equation of sphere is

$$\begin{aligned} (x+10)^2+\left(y-\frac{17}{3}\right)^2+\left(z+\frac{14}{3}\right)^2 &= \frac{1481}{9} \\ 9(x^2+y^2+z^2)+180x-102y+84z-96 &= 0. \end{aligned}$$

**Example 8.27** Find the equation of the sphere passing through the points $(2,0,0),(0,2,0),(0,0,2)$ and having the radius as small as possible.

**Solution:** Let the required equation of sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad (1)$$

Since $(2,0,0),(0,2,0),(0,0,2)$ lies on sphere (1). Therefore

$$\begin{aligned} 4+4u+d &= 0 \\ 4+4v+d &= 0 \\ 4+4w+d &= 0 \end{aligned}$$

This gives $d=-4-4u=-4-4v=-4-4w$
i.e. $u=v=w$ (2)
Radius of sphere is

$$\begin{aligned} r &= \sqrt{u^2+v^2+w^2-d} = \sqrt{u^2+u^2+u^2-(-4-4u)} \\ r &= \sqrt{3u^2+4u+4} \end{aligned}$$

For $r$ to be minimum $\dfrac{dr}{du}=0$ and $\dfrac{d^2r}{du^2}>0$.

$$\begin{aligned} \frac{dr}{du} &= \frac{1(6u+4)}{2\sqrt{3u^2+4u+4}}=0 \\ 6u+4=0 \quad \text{i.e.} \quad u &= -\frac{4}{6}=-\frac{2}{3}. \end{aligned}$$

We leave it as an exercise to show that $\dfrac{d^2r}{du^2}>0$. By (2) $u=v=w=\frac{-2}{3}$ and $d=-4+\frac{8}{3}=\frac{-4}{3}$. Using above value of $u,v,w$ and $d$ in (1)

$$3(x^2+y^2+z^2)-4x-4y-4z-4=0$$

This is required equation of sphere.

**Example 8.28** Discuss the position of point $P(0,1,2)$ with respect to the sphere $x^2+y^2+z^2-6x+4y-2z-11=0$.
**Solution.** The centre of given sphere $=C(-u,-v,-w)=(3,-2,1)$ and radius equals $r=\sqrt{u^2+v^2+w^2-d}=\sqrt{9+4+1+11}=\sqrt{25}=5$.
Since $P(0,1,2)$, by distance formula
$CP=\sqrt{(3-0)^2+(-2-1)^2+(1-2)^2}=\sqrt{9+9+1}=\sqrt{19}$
$CP=\sqrt{19}<5=\text{radius}$. Hence $P(0,1,2)$ lies inside the sphere.



**Example 8.29** Find the length of the intercept cut off by the line $x+2=y+3=z+5$ on the sphere $x^2+y^2+z^2-6x+5z+11=0$.
**Solution:** We have given equation of sphere

$$x^2+y^2+z^2-6x+5z+11=0 \qquad (1)$$

and equations of line $x+2=y+3=z+5=r$ say. If $P(x,y,z)=P(r-2,r-3,r-5)$ lies on (1) then

$$r^2-7r+12=0 \text{ i.e. } r=4, \text{ or } r=3.$$

When $r=4, P(x,y,z)=(2,1,-1)$, and when $r=3, Q(x,y,z)=(1,0,-2)$. Length of intercept cut off by line

$$PQ=\sqrt{(2-1)^2+(1-0)^2+(-1+2)^2}=\sqrt{1+1+1}=\sqrt{3}.$$

**Example 8.30** Find the value of $\lambda$ if the plane $x+y+z=\lambda$ touches the sphere $x^2+y^2+z^2+2x+2y-2z-9=0$. Also then find the point of touching.
**Solution:** The given equation of sphere is

$$x^2+y^2+z^2+2x+2y-2z-9=0 \qquad (1)$$

The centre of sphere (1) $=C(-1,-1,1)$ and radius of sphere
$=r=\sqrt{1+1+1+9}=\sqrt{12}=2\sqrt{3}$.
The plane $x+y+z=\lambda$ will touch the sphere (1) if the length of the perpendicular from $C(-1,-1,1)$ to the plane = radius of the sphere.

$$\begin{aligned} \left|\frac{-1-1+1-\lambda}{\sqrt{3}}\right| &= 2\sqrt{3} \\ |-1-\lambda| &= 6 \\ \lambda=5 \quad \text{or} \quad & \lambda=-7. \end{aligned}$$

The direction ratios of a normal to the plane $x+y+z=\lambda$ are $1,1,1$. Equations of line passing through $C(-1,-1,-1)$ and having d.r.s $1,1,1$ are $\dfrac{x+1}{1}=\dfrac{y+1}{1}=\dfrac{z-1}{1}=r$ say. Thus, $(x,y,z)=(r-1,r-1,r+1)$

is any point on the line.
If $\lambda = -7$, then the equation of the plane is $x + y + z = -7$ and the point $P(r-1, r-1, r+1)$ lies on this plane. Hence, $r = -2$ and the point of contact is $P(-3, -1, -1)$.

If $\lambda = 5$, then the equation of the plane is $x + y + z = 5$ and the point $Q(r-1, r-1, r+1)$ lies on this plane. Hence, $r = 2$ and the point of contact is $Q(1, 1, 3)$.

**Example 8.31** Show that the tangent planes at any common point of the two spheres

$$\begin{aligned} S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d &= 0 \text{ and} \\ S' = x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' &= 0 \end{aligned}$$

are at right angle if $2uu' + 2vv' + 2ww' = d + d'$.

Let $P(x_1, y_1, z_1)$ be any common point of the two spheres

$$\begin{aligned} S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d &= 0 \qquad (1) \\ S' = x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' &= 0 \qquad (2) \end{aligned}$$

Therefore

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d &= 0 \qquad (3) \\ x_1^2 + y_1^2 + z_1^2 + 2u'x_1 + 2v'y_1 + 2w'z_1 + d' &= 0 \qquad (4) \end{aligned}$$

The equations of tangent planes at $P(x_1, y_1, z_1)$ to the two sphere are

$$\begin{aligned} xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d &= 0 \\ xx_1 + yy_1 + zz_1 + u'(x + x_1) + v'(y + y_1) + w'(z + z_1) + d' &= 0 \end{aligned}$$

i. e. $(x_1 + u)x + (y_1 + v)y + (z_1 + w)z + ux_1 + vy_1 + wz_1 + d = 0$ and

$$(x_1 + u')x + (y_1 + v')y + (z_1 + w')z + u'x_1 + v'y_1 + w'z_1 + d' = 0.$$

These two planes are perpendicular if

$$(x_1 + u)(x_1 + u') + (y_1 + v)(y_1 + v') + (z_1 + w)(z_1 + w') = 0$$



$x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1 + u'x_1 + v'y_1 + w'z_1 + uu' + vv' + ww' = 0.$
Multiplying by 2
$2x_1^2 + 2y_1^2 + 2z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + 2u'x_1 + 2v'y_1 + 2w'z_1$
$+ 2uu' + 2vv' + 2ww' = 0$
$(x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1)$
$+ (x_1^2 + y_1^2 + z_1^2 + 2u'x_1 + 2v'y_1 + 2w'z_1) + 2uu' + 2vv' + 2ww' = 0.$
By using (3) and (4)

$$\begin{aligned} -d - d' + 2uu' + 2vv' + 2ww' &= 0 \\ 2uu' + 2vv' + 2ww' &= d + d'. \end{aligned}$$

**Example 8.32** Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x - 2y + 2z - 12 = 0$ which are parallel to the plane $4x + y + z = 5$. State their points of contact.

**Solution:** The given equation of sphere is

$$x^2 + y^2 + z^2 - 4x - 2y + 2z - 12 = 0 \qquad (1)$$

The centre of sphere (1) $= C(-2, 1, -1)$. The direction ratios of normal to the plane $4x + y + z = 5$ are $4, 1, 1$. Equations of line passing through $C(-2, 1, -1)$ and having d.r.s. $4, 1, 1$ are

$$\frac{x+2}{4} = \frac{y-1}{1} = \frac{z+1}{1} = r \text{ say}$$

Thus, $x = 4r - 2, y = r + 1, z = r - 1$. Using above value of $(x, y, z)$ in equation (1) $18r^2 - 18 = 0$ i.e. $r = \pm 1$.
When $r = 1, P(x, y, z) = (2, 2, 0)$ and when $r = -1, Q(x, y, z) = (-6, 0, -2)$. We know equation of tangent plane at point $(x_1, y_1, z_1)$ to the general equation of spheres

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$$

Hence, the equation of tangent plane at $P(2, 2, 0)$ to sphere (1) is

$$4x + y + z - 10 = 0.$$

Similarly equation of tangent plane at $Q(-6, 0, -2)$ to sphere (2) is $4x + y + z + 26 = 0$.

**Example 8.33** Find the equation of the sphere having the circle

$$x^2+y^2+z^2+10y-4z-8=0, x+y+z=3$$

as the great circle.

**Solution:** The equation of the sphere passing through given circle is of the form

$$x^2+y^2+z^2+10y-4z-8+\lambda(x+y+z-3)=0 \qquad (1)$$

The centre of sphere (1) is $\left(\frac{-\lambda}{2}, -5-\frac{\lambda}{2}, 2-\frac{\lambda}{2}\right)$. Since given circle is a great circle of required sphere (1), its centre will lie on the plane of the circle $x+y+z=3$

$$\frac{-\lambda}{2}-5-\frac{\lambda}{2}+2-\frac{\lambda}{2}=3 \text{ i.e. } \lambda=-4.$$

Putting $\lambda=-4$ in equation (1) we get

$$\begin{aligned}(x^2+y^2+z^2+10y-4z-8)-4(x+y+z-3) &= 0\\ x^2+y^2+z^2-4x+6y-8z+4 &= 0\end{aligned}$$

**Exercise**

1. Find the equation of the sphere

   (i) whose centre is at origin and radius 5.

   (ii) whose centre is at $(-1,2,1)$ and radius 3.

   (iii) passing the points $A(2,3,-1)$ and $B(1,1,0)$ and whose centre lies on $\frac{x}{3}=\frac{y+1}{-2}=\frac{z-2}{2}$.

   (iv) passing through the origin and making equal positive intercept 3 units of the axes.

   (v) which circumscribes the tetrahedron. $(0,0,0),(1,0,0),(0,2,0),(0,0,3)$.

   (vi) through the four points $A(4,-1,2), B(0,-2,3), C(1,-5,1)$ and $D(2,0,1)$.



   (vii) which passes through three points $(3,0,2)$, $(-1,1,1)$, $(2,-5,4)$ and having centre on the line $2x+3y+4z=6$.

2. Find the centre and radius of the following sphere
   (i) $x^2+y^2+z^2-2x-4y-6z+5=0$.
   (ii) $4(x^2+y^2+z^2)+6x+4y-10z-1=0$

3. Find the equation of the smallest sphere through $A(2,-3,4)$ and $B(-5,6,-7)$.

4. Find the equation of the sphere described on the join of the points $A(-1,2,3)$ and $B(1,3,-4)$ as a diameter.

5. Find the co-ordinates of the points in which the line

$$\frac{x-8}{4}=\frac{y}{1}=-(z-1)$$

   cuts the sphere $x^2+y^2+z^2-4x+6y-2z+5=0$.

6. Find the length of the intercept made by the line $\frac{x-7}{2}=\frac{y-6}{1}=\frac{z+5}{-1}$ with the sphere $x^2+y^2+z^2-2x+3y-5z-31=0$.

7. Find the co-ordinates of the centre and radius of the circle $x^2+y^2+z^2-2x-4y+2z-30=0,\ 2x-y+2z-7=0$.

8. Find the area of the circle cut off the sphere $x^2+y^2+z^2-4x+6z-3=0$ by the plane $x+2y-2z=17$.

9. Find the area of the circle cut of the sphere $x^2+y^2+z^2=16$ by the plane $-x+2y+2z=9$.

10. Find the equation of the sphere

    (i) through the circle $x^2+y^2+z^2=9, 2x+3y+4z=5$ and the point $(1,2,3)$.

    (ii) through the circle $x^2+y^2+z^2+6x-4y-6z-14=0$, $x+y-z=0$ and the point $(1,1,-1)$.

(iii) passing through the circle $x^2+y^2+z^2+2z+3y-6=0$, $x-2y+4z-9=0$ and through the centre of the sphere $x^2+y^2+z^2-2x-4y+6z+5=0$.

(iv) passing through the origin and the circle $x^2+y^2+z^2+2x+3y+2z-6=0, x+2y+4z+9=0$.

11. Find the equation of the sphere passing through the points $(1,0,0),(0,1,0),(0,0,1)$ and having the radius as small as possible.

12. Find the positions of the points $P(1,1,1), Q(-2,-2,-2)$ w.r.t. the sphere $x^2+y^2+z^2-6x+4y-2z-2=0$.

13. Discuss the position of a point $P(2,-3,0)$ w.r.t. the sphere $x^2+y^2+z^2+2x-4y-4z+8=0$.

14. Obtain the equation of the circle lying on the sphere $x^2+y^2+z^2-2x+4y-6z+3=0$ and having its centre at $(2,3,-4)$.

15. Find the centre and radius of the circle $x^2+y^2+z^2-4x+6z-3=0, x+2y-2z=17$.

16. Find the co-ordinates of the centre and radius of the circle $x^2+y^2+z^2-2x-4y+2z-30=0, 2x-y+2z-7=0$.

17. Find the angle between the tangent planes to the sphere $x^2+y^2+z^2+4x-6y=0$ at $(-2,6,2)$ and $(0,-3,3)$.

18. If the co-ordinates of points $A,B$ are $(2,2,-2)$, $(1,-1,1)$ respectively. Find the locus of $P$ such that $AP=\sqrt{2}BP$. Show that the locus is a sphere and find its centre and radius.

19. Show that the spheres $x^2+y^2+z^2-18x-24y-40z+225=0$ and $x^2+y^2+z^2=25$ touch each other and find the point of contact.

20. Show that the plane $2x-2y+z+16=0$ touches the sphere $x^2+y^2+z^2+2x-4y+2z-3=0$. Also find the point of contact.



21. Show that the two spheres $x^2+y^2+z^2-2x-6y-15=0$ and $10x^2+10y^2+10z^2-20x+52y+84z+214=0$ touch each other and find point of contact.

22. Show that the plane $lx+my+nz=p$ will touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ if $(ul+vm+wn+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$.

23. Find the equations of the two tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$.

24. Show that the spheres $x^2+y^2+z^2-24x-40y-18z+225=0$ and $x^2+y^2+z^2=25$ touch externally and find the point of contact.

25. Find the area of the section of a sphere with centre $(-6,1,2)$ and radius 4 which is cut by a plane $x-y+2z+5=0$.

26. Find the centre and radius of the section of the sphere $x^2+y^2+z^2-4x+4y-6z-8=0$ by the
(i) $xy$- plane (ii) $yz$- plane (iii) $z=8$ plane.

27. Find the value of $k$ for which the plane $x+y+z-k\sqrt{3}=0$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$.

28. Find the equation of the sphere having the circle $x^2+y^2+z^2-2x-2y-2z-22=0, x+2y+2z+7=0$ as the great circle.

29. Find the equation of the sphere on $AB$ as a diameter where $A(2,-3,1)$ and $B(-1,-2,4)$.

30. The sphere of constant radius $k$ passes through the origin and meets the axes in $A,B,C$. Prove that the centriod of the triangle $ABC$ lies on the sphere.

31. A tangent plane at a variable point $(\alpha,\beta,\gamma)$ on the sphere $x^2+y^2+z^2=a^2$ meets the axes in $ABC$. Show that the locus of the centre of the sphere $OABC$ is $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{4}{a^2}$.

## Answers

1. (i) $x^2 + y^2 + z^2 = 25$
(ii) $x^2 + y^2 + z^2 + 2x - 4y - 2z - 3 = 0.$
(iii) $9(x^2 + y^2 + z^2) + 180x - 102y + 84z - 96 = 0.$
(iv) $x^2 + y^2 + z^2 - 2x - 3y - 3z = 0.$
(v) $x^2 + y^2 + z^2 - x - 2y - 3z = 0.$
(vi) $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0.$
(vii) $x^2 + y^2 + z^2 + 4y - 6z - 1 = 0.$

2. (i) Centre $(1, 2, 3)$ radius $= 3$. (ii) Centre $\left(\frac{-3}{4}, \frac{-2}{4}, \frac{5}{4}\right)$

3. $x^2 + y^2 + z^2 + 3x - 3y + 3z - 56 = 0.$

4. $x^2 + y^2 + z^2 - 5y + z - 7 = 0.$

5. $(0, -2, 3),\ (4, -1, 2)$. 6. $\sqrt{54}$. 7. $(3, 1, 1), 3\sqrt{3}$

8. $7\pi$ 9. $7\pi$

10. (i) $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$
(ii) $x^2 + y^2 + z^2 - 4x + 6z - 3 = 0, y + z + 1 = 0.$
(iii) $3(x^2 + y^2 + z^2) + 8x + 5y + 8z - 36 = 0.$
(iv) $3(x^2 + y^2 + z^2) + 8x + 13y + 14z = 0.$

11. $3x^2 + 3y^2 + 3z^2 - 2x - 2y - 2z - 1 = 0.$

12. $P$ inside, $Q$ outside.

13. $P$ outside the sphere.

14. $x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0, x + 5y - 7z - 45 = 0.$

15. $(3, 2, -5), \sqrt{7}$ 16. $(3, 1, 1), 3\sqrt{3}$

17. $\cos\theta = \frac{6}{7\sqrt{13}}$

18. Centre $(0, -4, 4)$, radius $= \sqrt{38}$

19. $\left(\frac{9}{5}, \frac{12}{5}, 4\right)$ 20. $(-3, 4, -2)$ 21. $(1, -1, -3)$



23. $2x + 2y - z + 10 = 0, 2x + 2y - z - 8 = 0.$

24. $\left(\frac{12}{5}, \frac{20}{5}, \frac{9}{5}\right)$ 25. $\frac{40\pi}{3}$

26. (i) $(2, -2, 0), 4$ (ii) $(0, -2, 3), \sqrt{21}$ (iii) $(2, -2, 8), r = 0$

27. $\sqrt{3} \pm 3$

28. $3(x^2 + y^2 + z^2) + 2x + 10y + 10z - 10 = 0.$

29. $x^2 + y^2 + z^2 - x + 5y - 5z + 8 = 0.$

# Chapter 9

# Cones and Cylinders

## 9.1 Cone

**Definition 9.1** A surface generated by a straight line passing through a fixed point and intersecting a given curve is called as $a\ cone$.
The fixed point is called the $vertex\ of\ the\ cone$ and the given curve is called the $guiding\ curve$. A line which generates the cone is called $a\ generator$.

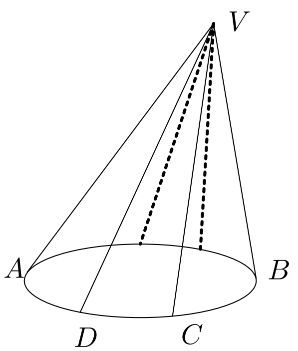



Figure 9.1: Cone with a guiding curve

The surface in the Fig. 9.1 is a cone with vertex $V$, The line $VA$ as a generator. The lines $VB, VC$ are also generators. In fact every line joining $V$ to any point of the guiding curve is a generator of the cone.

**Remark 9.1** If the guiding curve is a plane curve of degree $n$, then the equation of the cone is also of degree $n$ and we call it $a\ cone\ of\ order\ n$.





In this chapter, we study only quadratic cones; i.e. cones having its equation of second degree in $x, y$ and $z$.

## 9.2 Equation of a cone

To find the equation of a cone with vertex $V(\alpha, \beta, \gamma)$ and whose guiding curve is the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0. \tag{1}$$

We have to find the locus of points on lines which pass through the vertex $V(\alpha, \beta, \gamma)$ and intersects the given guiding curve. Observe that the equations of any line passing through the vertex $V(\alpha, \beta, \gamma)$ and having direction ratios $l, m, n$ is given by,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \tag{2}$$

Any point on the equation (2) has the coordinates, $(\alpha+lt, \beta+mt, \gamma+nt)$, where $t \in \mathbb{R}$. If the line (2) intersects the guiding curve given by (1), then we have,

$$\begin{aligned} a(\alpha+lt)^2 + 2h(\alpha+lt)(\beta+mt) + b(\beta+mt)^2 + 2g(\alpha+lt) \\ +2f(\beta+mt) + c = 0, (\gamma+nt) = 0 \end{aligned} \tag{3}$$

Eliminating $t, l, m$ and $n$ between (2) and (3), we get

$$\begin{aligned} a(\alpha z+\gamma x)^2 + 2h(\alpha z+\gamma x)(\beta z+\gamma y) + b(\beta z+\gamma y)^2 & \\ +2g(\alpha z+\gamma x)(z-y) + 2f(\beta z+\gamma y)(z-y) + c(z-y)^2 &= 0. \end{aligned} \tag{4}$$

This is the required equation of the cone.

**Remark 9.2** *From the equation* (4) *it can be seen that the equation of a quadratic cone is of second degree in x, y and z; i.e.,*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \tag{5}$$

**Example 9.1** *Find the equation of a cone with vertex at the point (3,1,2) and guiding curve is* $2x^2 + 3y^2 = 1, z = 0$ .

**Solution** The vertex is $V(3,1,2)$. Let $a, b, c$ be the direction ratios of a generator of the cone. Then the equations of generator are,

$$\frac{x-3}{a} = \frac{y-1}{b} = \frac{z-2}{c} = t(say) \tag{6}$$

The coordinates of any point on the generator are $(3+at, 1+bt, 2+ct)$. For some $t \in \mathbb{R}$, $(3+at, 1+bt, 2+ct)$ lies on the guiding curve. Therefore $2(3+at)^2 + 3(1+bt)^2 = 1$ and $2 + ct = 0$. Thus, $t = \frac{-2}{c}$. From this we get,

$$2\left(3 - \frac{2a}{c}\right)^2 + 3\left(1 - \frac{2b}{c}\right)^2 = 1$$

From (6), we obtain $2[3 - 2(\frac{x-3}{z-2})]^2 + 3[1 - 2(\frac{y-1}{z-2})]^2 = 1$ so $2[3(z-2) - 2(x-3)]^2 + 3[1(z-2) - 2(y-1)]^2 = (z-2)^2$.

After simplification, the required equation of the cone is,

$$2x^2 + 3y^2 + 5z^2 - 3yz - 6xz + z - 1 = 0.$$

## 9.3 Cone with vertex at the origin

Recall that the general equation of second degree equation is,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

If any one of the constants $u, v, w$ and $d$ is non-zero, then the equation is non-homogeneous in $x, y$ and $z$. If each of the constants $u, v, w$ and $d$, is zero, then the resulting equation, $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$, is a homogeneous second degree equation in $x, y$ and $z$.

**Theorem 9.1** The equation of a cone with vertex at the origin is a homogeneous second degree equation.

**Proof.** Let the equation of a quadratic cone with vertex at the origin be,



$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \tag{7}$$

We will show that $u = v = w = d = 0$. Let $P(x', y', z')$ be any point on the cone given by (7). Since $O(0,0,0)$ is the vertex, the line $OP$ is a generator of the cone. The equation of the line $OP$ is given by,

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = t(say)$$

Thus the coordinates of any point on the line $OP$ are $(tx', ty', tz')$. Since $OP$ is a generator, these coordinates satisfy the equation (7),

$\therefore a(tx')^2 + b(ty')^2 + c(tz')^2 + 2f(ty')(tz')$
$+ 2g(tz')(tx') + 2h(tx')(ty') + 2u(tx') + 2v(ty') + 2w(tz') + d = 0.$

This equation can be written as,

$$\begin{aligned}(ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y')t^2 \\ +2(ux' + 2vy' + 2wz')t + d = 0\end{aligned} \tag{8}$$

Observe that (8) is a quadratic equation for $\forall\ t \in \mathbb{R}$. This is possible only if each of the coefficient is $0$. Therefore,

$$\begin{aligned} ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' &= 0, \\ ux' + 2vy' + 2wz' = 0 \text{ and } \quad d &= 0. \end{aligned}$$

We now claim that each of the constants $u, v$ and $w$ are zero. For if not i.e. if at least one of the constants is not zero, then the equation $ux + vy + wz = 0$ would represent a plane and the point $(x', y', z')$ lies on the plane. As the point $(x', y', z')$ lies on the surface (7) , it would mean that the surface (7) is a plane. This is impossible as it represents a cone. Hence $u = 0, v = 0, w = 0$ and we also have $d = 0$. So that the equation of a cone with vertex at the origin is of the form,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

which is a homogeneous in $x, y$ and $z$. ∎

Note that the converse of this theorem is also true.

**Theorem 9.2** Every second degree homogeneous equation in $x, y, z$ represents a cone with vertex at the origin.
**Proof.** Consider the homogeneous equation of second degree in $x, y, z$

$$i.e. \quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \tag{9}$$

The coordinates of the origin satisfy the equation (10). Thus, the origin $O$ lies on the surface given by (9). Let $P(x', y', z')$ be any other point on the surface given by (10).

$$\therefore ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0 \tag{10}$$

Now we show that the line $OP$ lies on the surface given by (9). The equation of the line $OP$ is,

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = t(say)$$

The coordinates of any point on the line $OP$ are $(tx', ty', tz')$. Thus substituting these coordinates in $L.H.S.$ of (9) we have,

$$\begin{aligned}
& a(tx')^2 + b(ty')^2 + c(tz')^2 + 2f(ty')(tz') + 2g(tz')(tx') + 2h(tx')(ty') \\
= \;& t^2(ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y') \\
= \;& t^2(0) \quad \ldots (9) \\
= \;& 0.
\end{aligned}$$

Hence, the point with coordinates $(tx', ty', tz')$ satisfies (9). Therefore any point on the line $OP$ lies on the surface given by (9). Thus (9) is the surface generates by $OP$. Therefore the equation (9) represents a cone with vertex at the origin.

**Remark 9.3** If the line with equation, $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator of a cone with vertex at the origin, then the direction ratios $l, m, n$ satisfies the



equation of the cone.
Let the equation of the cone be,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

The coordinates of any point on the line are $(lt, mt, nt)$, where $t \in \mathbb{R}$. These coordinates satisfy the equation.

$$\begin{aligned}
\therefore \quad & a(lt)^2 + b(mt)^2 + c(nt)^2 + 2f(mt)(nt) \\
& +2g(nt)(lt) + 2h(lt)(mt) = 0 \\
\therefore \quad & t^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) = 0.
\end{aligned}$$

This equation holds of all values of $t$.

$$\therefore al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0.$$

Thus, it follows that the direction ratios of a generator satisfy the equation of the cone.

Also, it is easy to see that if the direction ratios $l, m, n$ of a generator of a cone with vertex at the origin satisfy the equation $\phi(l, m, n) = 0$, then the equation of the cone is $\phi(x, y, z) = 0$. ∎

**Remark 9.4** The general equation of a quadratic cone with vertex at the origin is,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Let the origin be shifted to the point $V(\alpha, \beta, \gamma)$. In the new coordinate system the above equation becomes,

$$\begin{aligned}
& a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 \\
+ \quad & 2f(y-\beta)(z-\gamma) + 2g(z-\gamma)(x-\alpha) + 2h(x-\alpha)(y-\beta) = 0.
\end{aligned}$$

This is a general equation of a quadratic cone with vertex at $(\alpha, \beta, \gamma)$.

## 9.4 The Right circular Cone

**Definition 9.2** A right circular cone is a surface generated by a straight line passing through a fixed point and making a constant angle $\theta$ with a fixed straight line passing through the given point. The fixed point is called as $the\ vertex$ of the right circular cone, the fixed straight line is called as $the\ axis$ of the cone and the constant angle $\theta$ is called as $the\ semi-vertical\ angle$ of the cone.

**Remark 9.5** Every section of a right circular cone by a plane perpendicular to its axis is a circle.

Let $\theta$ be the semi - vertical angle of the cone and $\alpha$ be a plane perpendicular to the axis $VN$ see Fig. 9.2 of the cone. We show that the section of the cone by the plane $\alpha$ is a circle.

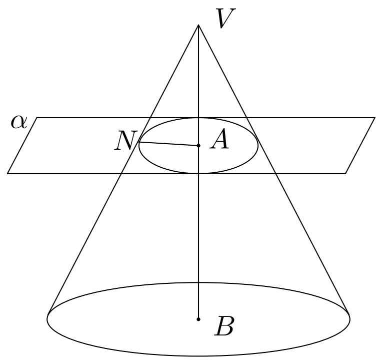



Figure 9.2: Section of a cone with a plane

Let $P$ be any point of the section of the cone by the plane $\alpha$. Let $A$ be the point of intersection of the axis $VN$ and the plane $\alpha$. Then $AP$ is perpendicular to $VA$. In the right angled triangle $VAP$, $\tan\theta = \frac{AP}{AV}$. $\therefore AP = AV\tan\theta$. Since, $AV$ and $\tan\theta$ are constant. it follows that $AP$ is constant for all points $P$ on the section of the cone by the plane $\alpha$. Thus the section is a circle.



### 9.4.1 Equation of a right circular cone

To find the equation of a right circular cone with a vertex $V(\alpha.\beta,\gamma)$ and whose axis is the line, $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, with a semi-vertical angle $\theta$. Let $VN$ be the axis of the cone and let $P(x, y, z)$ be any point on the cone. The direction rations of the generator $VP$ are $x-\alpha, y-\beta, z-\gamma$.

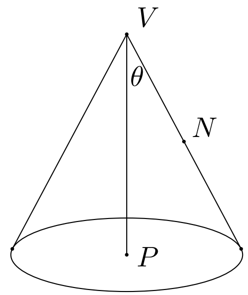



Figure 9.3:

Let $P$ be any point of the section of the cone by the plane $\alpha$. Let $A$ be the point of intersection of the axis $VN$ and the plane $\alpha$. Then $AP$ is perpendicular to $VA$. In the right angled triangle $VAP$, $\tan\theta = \frac{AP}{AV}$. $\therefore AP = AV\tan\theta$. Since, $AV$ and $\tan\theta$ are constant. it follows that $AP$ is constant for all points $P$ on the section of the cone by the plane $\alpha$. Thus the section is a circle.

The angle between $VP$ and $VN$ is $\theta(see\ Fig.\ 9.3)$. As the direction ratios of the axis $VN$ are $l, m, n$, we have,

$$\cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Therefore, the equation of the right circular cone is,

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+m^2+n^2)\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]\cos^2\theta$$

The following example will make the above proof more clear.

**Example 9.2** Find the equation of the right circular cone with vertex at $(2, -1, 4)$, the line $\dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z-4}{-1}$ as the axis and semi-vertical angle $\cos^{-1}(4/\sqrt{6})$.

**Solution.** Let $P(x, y, z)$ be any point on the cone with vertex $V(2, -1, 4)$. Then the direction ratios of a generator $VP$ are $x-2, y+1, z-4$. The direction ratios of the axis $\dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z-4}{-1}$ are $1, 2, -1$. Let $\theta$ be the semi-vertical angle. Hence, $\cos\theta = \frac{4}{\sqrt{6}}$. From $Remark$, we have,

$$\cos\theta = \frac{1(x-2) + 2(y+1) + (-1)(z-4)}{\sqrt{1+4+1}\sqrt{(x-2)^2 + (y+1)^2 + (z-4)^2}}$$

$$\therefore \quad \frac{4}{\sqrt{6}} = \frac{x + 2y - z + 4}{\sqrt{6}\sqrt{(x-2)^2 + (y+1)^2 + (z-4)^2}}$$

Hence, the required equation of the right circular cone is,

$$16\left[(x-2)^2 + (y+1)^2 + (z-4)^2\right] = (x + 2y - z + 4)^2$$

i.e.

$$15x^2 + 12y^2 + 15z^2 - 4yz - 2zx = 4xy - 80x + 16y - 120z + 320 = 0$$

## 9.5 Cylinders

**Definition 9.3** A surface generated by a straight line which always remains parallel to the given fixed line and which intersects to the given curve is called $a\ cylinder$. The straight lines which generate the cylinder is called as *the generators* of the cylinder and the given curve is called as *the guiding curve.*

## 9.6 Equation of a cylinder

To find the equation of the cylinder whose generator intersects the conic, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,\ z = 0$ and are parallel to the line, $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$.



Let $P(x, y, z)$ be any point on the cylinder. Then generator through $P$ is parallel to the line $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$. Hence the $d.r.s$ of the generator are $l, m, n$. The equation of the generator through $P$ is,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

The point of intersection of the generator and the plane $z = 0$ is given by,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{0 - z_1}{n}.$$

$$\therefore \quad \frac{x - x_1}{l} = \frac{-z_1}{n} \quad and \quad \frac{y - y_1}{l} = \frac{-z_1}{n}.$$

$$\therefore \quad x = x_1 - \frac{lz_1}{n}, \quad y = y_1 - \frac{mz_1}{n}.$$

Therefore the coordinates of the point of intersection of the generator and the plane $z = 0$ are, $(x_1 - \dfrac{lz_1}{n}, y_1 - \dfrac{mz_1}{n}, 0)$. But this point lies on the guiding curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

$$\begin{aligned}\therefore \quad a(x_1 - \frac{lz_1}{n})^2 + 2h(x_1 - \frac{lz_1}{n})(y_1 - \frac{mz_1}{n}) + b(y_1 - \frac{mz_1}{n})^2 \\ +2g(x_1 - \frac{lz_1}{n}) + 2f(y_1 - \frac{mz_1}{n}) + c = 0\end{aligned}$$

Hence, the locus of P is,

$$\begin{aligned}a(x - \frac{lz}{n})^2 + 2h(x - \frac{lz}{n})(y - \frac{mz}{n}) + b(y - \frac{mz}{n})^2 \\ +2g(x - \frac{lz}{n}) + 2f(y - \frac{mz}{n}) + c = 0\end{aligned}$$

which when simplified, gives the equation of the cylinder as,

$$\begin{aligned}a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 \\ +2gn(nx - lz) + 2fn(ny - mz) + cn^2 = 0\end{aligned}$$

**Example 9.3** Find the equation of a cylinder whose generators are parallel to the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2 = 1$ and $z = 0$.
**Solution.** Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then the generator through $P$ is parallel to the line, $\frac{x}{2} = \frac{y}{1} = \frac{z}{3}$. Therefore the equation of the generator through $P$ are,

$$\frac{x-x_1}{2} = \frac{y-y_1}{1} = \frac{z-z_1}{3} = t(say)$$

Coordinates of any point on this line are $(x_1+2t, y_1+t, z_1+3t)$. For some $t$, this point lies on guiding curve $x^2 + 2y^2 = 1$, $z = 0$.

$$\therefore \quad (x_1 + 2t)^2 + 2(y_1 + t)^2 = 1 \tag{11}$$

also,

$$z_1 + 3t = 0, \therefore \quad t = \frac{-z_1}{3}.$$

Substituting the value of $t$ in (11), we get

$$(x_1 - 2\frac{z_1}{3})^2 + 2(y_1 - \frac{z_1}{3})^2 = 1$$

$$\therefore (3x_1 - 2z_1)^2 + 2(3y_1 - z_1)^2 = 9$$

Hence, locus of $P$ is, $(3x - 2z)^2 + 2(3y - z)^2 = 9$.

$$i.e. 9x^2 + 18y^2 + 6y^2 - 12xz - 12yz - 9 = 0,$$
$$i.e. 3x^2 + 6y^2 + 2y^2 - 4xz - 4yz - 3 = 0$$

## 9.7 Right circular cylinder

**Definition 9.4** A cylinder is called a $right\ circular\ cylinder$ if its guiding curve is a circle and its generators are lines perpendicular to the plane containing the circle. The normal to the plane of the guiding circle passing through its centre is called as the $axis\ of\ the\ cylinder$. If we take a section of the cylinder by a plane perpendicular to the axis of the cylinder, then this section will be a circle. The radius of this circle is called as the $radius\ of\ the\ cylinder$.



### 9.7.1 Equation of a right circular cylinder

To find the equation of the right circular cylinder whose axis is the line $L : \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and radius is $r$. The point $A(\alpha, \beta, \gamma)$ lies on the line $L$ whose d.r.s. are $l, m, n$. Let $P(x, y, z)$ be a point on the cylinder $see\ Fig.\ 9.4$
Draw $PM$ perpendicular to the axis of the cylinder. Then $PM = r$. Now,

$$\begin{aligned} AP^2 &= (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 \\ MA &= projection\ \ of\ \ AP\ \ on\ \ the\ \ axis \\ \therefore MA &= \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}} \end{aligned}$$

Now, from the right angled $\triangle AMP$ , we get

$$\begin{aligned} AP^2 - MA^2 &= PM^2; \\ (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 & \\ - \left(\frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}\right)^2 &= r^2 \end{aligned}$$

which is the required equation of the cylinder.

**Example 9.4** Find the equation of the circular cylinder of radius 3 and axis passing through $(2, -1, 3)$ and having direction cosines proportional to $1, 2, -2$.
**Solution.** Let $A(2, -1, 3)$ and $P(x, y, z)$ be any point on the cylinder. Draw $PM$ perpendicular to the axis of the cylinder. Then $PM = 3$. By the distance formula,

$$\begin{aligned} AP^2 &= (x-2)^2 + (y+1)^2 + (z-3)^2 \\ MA &= projection\ \ of\ \ AP\ \ on\ \ the\ \ axis \\ \therefore \quad MA &= \frac{1(x-2) + 2(y+1) - 2(z-3)}{\sqrt{1^2+2^2+(-2)^2}} \\ MA &= \frac{x+2y-2y+6}{3} \end{aligned}$$

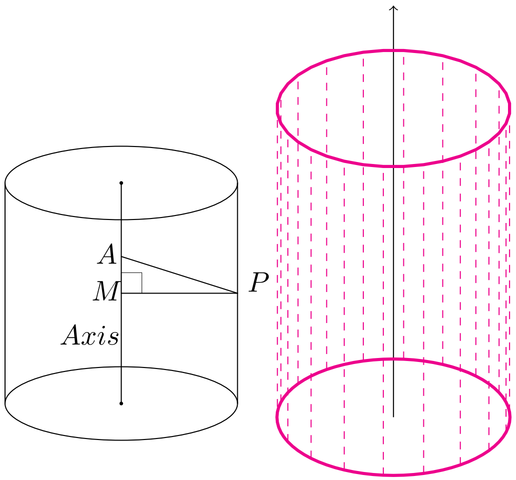



Figure 9.4:

Now, from the right angled triangle $\triangle AMP$, we have $AP^2 - MA^2 = 9$. Substituting the value of $AP$ and $MA$, we get

$$8x^2 + 5y^2 + 5z^2 + 8yz + 4zx - 4xy - 48x - 6y - 30z + 9 = 0,$$

which is the required equation of the cylinder.

## 9.8 Illustrative Examples

**Example 9.5** Find the general equation of the quadratic cone with the vertex at the origin and passing through the three coordinate axes.
**Solution.** The vertex of the cone is the origin.Hence its equation is a homogenous equation of degree 2 in $x, y, z$. Let the equation of the cone be,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad (12)$$

The cone given by (12) passes through the three co-ordinate axes. Therefore $x, y$ and $z$ axes are the generators of the cone. The direction ratios of



$x, y$ and $z$ axes satisfy the equation (12). $d.r.s$of the $X - axis$ are $1, 0, 0$; $d.r.s$ of the $Y - axis$ are $0, 1, 0$ and $d.r.s$ of the $Z - axis$ are $0, 0, 1$. From these conditions we get $a = 0, b = 0$ and $c = 0$. Substituting $a = 0, b = 0$ and $c = 0$ in (12), we get the general equation of the cone which passes through the three co-ordinate axes as

$$2fyz + 2gzx + 2hxy = 0 \text{ i.e. } fyz + gzx + hxy = 0.$$

**Example 9.6** Find the equation of the cone which passes through the axes of co-ordinates and contains the points $(1, 1, 1)$ and $(1, -2, 1)$.
**Solution.** As the cone passes through the three co-ordinate axes , the vertex of the cone is at the origin.Hence the equation of the cone is of the form'

$$fyz + gzx + hxy = 0 \qquad (13)$$

The cone passes through the points $(1, 1, 1)$ and $(-1, 2, 1)$. Co-ordinates of these two points satisfy the equation (13).

$$\therefore \quad f + g + h = 0 \quad and \quad 3f - g - 2h = 0$$

Solving the equations for $f, g$ and $h$ ,we get

$$\frac{f}{-1} = \frac{g}{4} = \frac{h}{-3} = k(say)$$
$$\therefore \quad f = -k, \quad g = 4k, \quad h = -3k.$$

Substituting the values of $f, g$ and $h$ in (13) we get the equation of the required cone as,

$$-kyz + 4kzx - 3kxy = 0 \text{ i.e. } yz - 4zx + 3xy = 0.$$

**Example 9.7** Find the equation of the cone passing through the co-ordinate axes and having the lines, $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x}{3} = \frac{y}{-1} = \frac{z}{-1}$ as generators.
**Solution.** As the cone passes through the three co-ordinate axes, the vertex of the cone is at the origin. Hence the equation of the cone is of the form

$$fyz + gzx + hxy = 0 \qquad (14)$$

Given that the lines $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ *and* $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ are the generators of the cone, thus $d.r.s$ of the lines satisfy the equation $(8.8.3)$

$$\begin{aligned} \therefore \quad f(-2)(3)+g(3)(1)+h(1)(-2)&=0 \\ and \quad f(-1)(-1)+g(-1)(3)+h(3)(-1)&=0 \end{aligned}$$

$$\therefore \quad -6f+3g-2h=0 \quad and \quad f-3g-3h=0.$$

Solving these two equations for $f, g$ and $h$ we get

$$\frac{f}{-1}=\frac{g}{4}=\frac{h}{-3}=k(say)$$
$$\therefore \quad f=-15k, \quad g=-20k, \quad h=15k.$$

Substituting the values of $f, g$ and $h$ in $(8.8.3)$, we get the equation of the required cone as

$$-15kyz-20kzx+15kxy=0 \text{ i.e. } \quad 3yz+4zx+3xy=0.$$

**Example 9.8** Show that the line $\frac{x}{2}=\frac{y}{-1}=\frac{z}{3}$ is a generator of the cone $x^2+y^2+z^2+4xy-xz=0$.
**Solution.** The equation of the cone is,

$$x^2+y^2+z^2+4xy-xz=0 \tag{15}$$

which is a homogenous equation. Hence the vertex of the cone is the origin. If we show that $d.r.s$ of the line $\frac{x}{2}=\frac{y}{-1}=\frac{z}{3}$ satisfy the equation (15), then we can say that the given line is a generator of the cone given by the equation (15). $d.r.s$ of the line $\frac{x}{2}=\frac{y}{-1}=\frac{z}{3}$ are $2,-1,3$. Substituting these values in $L.H.S.$ of (15) we get,

$$2^2+(-1)^2+3^2+4(2)(-1)-(2)(3).$$

Which is equal to $0$. Thus, the line $\frac{x}{2}=\frac{y}{-1}=\frac{z}{3}$ is the generator of the given cone.



**Example 9.9** Show that the equation

$$4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$$

represents a cone with with vertex at the point $(-1,-2,-3)$
**Solution.**

$$4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0 \tag{16}$$

Shift the origin to the point $(-1,-2,-3)$. Let $(x,y,z)$ and $(x',y',z')$ be respectively the old and new co-ordinates of the points on the cone. Then $x=x'-1, y=y'-1, z=z'-1$ Substituting in to the equation (16) we get

$$\begin{aligned} 4(x'-1)^2-(y'-2)^2+2(z'-3)^2+2(x'-1)(y'-2) \\ -3(y'-2)(z'-3)+12(x'-1)-11(y'-2)+6(z'-3)+4=0 \end{aligned}$$

On simplification gives

$$4x'^2-y'^2+2z'^2+2x'y'-3y'z'=0 \tag{17}$$

The equation (17) is a homogeneous equation in $x', y', z'$. Hence the equation (17) represents a cone with vertex at origin in the new co-ordinate system. Thus the equation (16) represents a cone with vertex at $(-1,-2,-3)$.

**Example 9.10** Find the equation of the cone whose vertex is at origin and the guiding curve is a circle $y^2+z^2=16, x=2$. Show that section of the cone by the plane $z=1$ is a hyperbola.
**Solution.** Since the vertex of the cone is at origin, the equation of the cone is a homogeneous equation in $x, y, z$. Consider the equation of the guiding curve $y^2+z^2=16, x=2$. We make one of the equations homogeneous with the help of the other. We make $y^2+z^2=16$ homogeneous with the help of $x=2$

$$\therefore \quad y^2+z^2=16\times 1^2 \therefore \quad y^2+z^2=16\times\left(\frac{x}{2}\right)^2 \quad , since \quad \frac{x}{2}=1$$

Hence the equation of the cone is $4x^2 - y^2 - z^2 = 0$. Now the section of the cone by the plane $z = 1$ is,

$$4x^2 - y^2 - 1^2 = 0, \quad z = 1 \text{ i.e. } \quad 4x^2 - y^2 = 1, \quad z = 1$$

which is a hyperbola.

**Example 9.11** Find the equation of the cone with its vertex at the origin and whose guiding curve is given by $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$, $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$.
**Solution.** Since the vertex of the cone is at origin, the equation of the cone is a homogeneous equation in $x, y, z$. The equation of the cone is obtained by making the equation

$$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0 \tag{18}$$

homogeneous with the help of the equation

$$x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0 \tag{19}$$

Subtracting the equation (18) from the equation (19) we get

$$4x + 2y + 2z - 8 = 0 \quad i.e. \quad \frac{2x + y + z}{4} = 1 \tag{20}$$

using (20) we make the equation (17) homogeneous as follows,

$$x^2 + y^2 + z^2 - 2x\left(\frac{2x+y+z}{4}\right)$$
$$+2y\left(\frac{2x+y+z}{4}\right) + 4z\left(\frac{2x+y+z}{4}\right) - 3\left(\frac{2x+y+z}{4}\right) = 0$$

Simplification yields,

$$12x^2 - 21y^2 - 29z^2 - 18yz - 12zx + 4xy = 0$$

which is the required equation of the cone.



**Example 9.12** Find the equation of the right circular cone which passes through the point $(1, -2, 3)$ whose vertex is at $(2, -3, 5)$ and whose axis makes equal angles with co-ordinate axes.
**Solution.** We are given that axis of the right circular cone makes equal angles with co-ordinate axes .Therefore the $d.r.s$ of the axis are $1, 1, 1$. Let $A(1, -2, 3)$ and $V(2, -3, 5)$ . Therefore the $d.r.s$ of $VA$ are $2 - 1, -3 + 2, 5 - 3$ i.e.$1, -1$ and $2$ respectively. Let $\theta$ be the semi vertical angle. The $\theta$ is the angle between the axis of the cone and $VA$

$$\therefore \quad \cos\theta = \frac{1(1) + 1(-1) + 1(2)}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{2}{\sqrt{3}\sqrt{6}} \tag{21}$$

Let $P(x, y, z)$ be a point on the right circular cone.Then $VP$ is a generator and its $d.r.s$ are $x - 2, y + 3, z - 5$. Then $\theta$ is the angle between $VP$ and the axis

$$\therefore \quad \cos\theta = \frac{1(x-2) + 1(y+3) + 1(z-5)}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{(x-2)^2 + (y-3)^2 + (z-5)^2}} \tag{22}$$

From (21) and (22) we get

$$\frac{2}{\sqrt{3}\sqrt{6}} = \frac{x + y + z - 4}{\sqrt{3}\sqrt{(x-2)^2 + (y-3)^2 + (z-5)^2}}$$
$$i.e. \quad 2[(x-2)^2 + (y-3)^2 + (z-5)^2]^2 = 3(x + y + z - 4)^2,$$

which is the required equation of the right circular cone.

**Example 9.13** Find the equation of the right circular cylinder of radius $2$, whose axis passes through $(1, 2, 3)$ and has $d.c.s$ proportional to $2, -3, 6$.
**Solution.** Let $A(1, 2, 3)$ and $P(x, y, z)$ be a point on the cylinder. Draw $PM$ perpendicular to the axis of the cylinder. Then $PM$ is the radius of the cylinder so $PM = 2$.. By Distance formula, $AP^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$. Let $MA$ pbe the projection of $AP$ on the axis. $\therefore$ $MA = \frac{2(x-1) - 3(y-2) + 6(z-3)}{\sqrt{2^2 + (-3)^2 + 6^2}}$ as d.r.s of the axis are $2, -3, 6$. Thus, $MA = \dfrac{2x - 3y + 6z - 14}{7}$.

Now from the right angled $\triangle AMP$, we get $AP^2 - MA^2 = 9$

$$\left((x-1)^2 + (y-2)^2 + (z-3)^2\right) - \left(\frac{2x-3y+6z-14}{7}\right)^2 = 9$$

Simplification of the equation gives the required equation of the right circular cylinder as

$$45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0.$$

## 9.9 Exercise

1. Find the equation of a cone whose vertex is at $(-1,1,2)$ and guiding curve is $3x^2-y^2=1; z=0$.

2. Find the equation of a cone with vertex at the origin and containing the curve $x^2+y^2=4; z=5$.

3. Find the equation of a cone whose vertex is at $(1,1,3)$ and passing through $4x^2+z^2=1; y=4$.

4. The axis of a right circular cone with vertex at the origin makes equal angles with the coordinate axes. If the cone passes through the line drawn from the origin with direction rations $1,-2,2$, find the equation of the cone.

5. Find the equation of the cylinder whose generators are parallel to the line $6x=-3y=2z$ and whose guiding curve is the ellipse $x^2+2y^2=1; z=3$.

6. Lines are drawn parallel to the line $\frac{x-3}{l}=\frac{y-4}{m}=\frac{z-5}{n}$ through the points on the circle $x^2+y^2=a^2$ in $ZOX-plane$. Find the equation of the surface so formed.

7. Find the equation of the right circular cylinder of radius 2 and having as axis the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$.



8. Find the equation of the right circular cone having $P(2,-3,5)$ as a vertex; axis PQ which makes equal angles with coordinate axes and the semi vertical angle is $30^o$

9. Show that $x^2+2y^2+z^2-4yz-6zx-2x+8y-2z+9=0$ represents a cone with vertex at $(1,-2,0)$.

10. Find the equation of a cone with vertex the origin and base a circle in the plane $z=12$ with centre $(13,0,12)$ and radius 5.Also show that the section of any plane parallel to $x=0$ is a circle.

11. Find the equation of the right circular cylinder of radius 2 whose axis passes through$(1,2,3)$ and has d.r.s. $2,-3,6$.

12. Find the equation of a cone whose vertex is at the origin and direction ratios of whose generators satisfy the equation $3l^2-2m^2+5n^2=0$.

13. The equation of a cone is $x^2+2y^2+z^2-2yz+zx-3xy=0$. Test whether the following lines are generators of the cone.
(a) $x=-y=z$ (b) $x=y=z$ (c) $\frac{x}{2}=\frac{y}{3}=\frac{z}{2}$ (d) $\frac{x}{3}=\frac{y}{-1}=\frac{z}{2}$.

14. Find the equation of a cone with vertex at the origin and which passes through the curve
$$\begin{aligned} x^2+y^2+z^2+x-2y+3z &= 4 \\ x^2+y^2+z^2+2x-3y+4z &= 5. \end{aligned}$$

15. Find the equation of the right circular cylinder of radius 2 whose axis lies along the line $\dfrac{x-1}{2}=\dfrac{y+3}{-1}=\dfrac{z-2}{5}$.

16. Obtain the equation of the right circular cylinder whose guiding curve is the circle $x^2+y^2+z^2-9=0; x-y+z-3=0$.

17. Lines are drawn through the origin having direction ratios $1,2,2$; $2,3,6$; and $3,4,12$. Show that the axis of the right circular cone through them has d.c.s.$\frac{-1}{\sqrt{3}}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}$ and the semi vertical angle of the cone is —-. Also obtain the equation of the cone.

18. Show that the equation $2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0$ represents a cone whose vertex is $(\frac{-7}{6}, \frac{1}{3}, \frac{5}{6})$.

19. Determine the equation of the right circular cone having vertex at $(2, 3, 1)$, axis parallel to the line $2x = -y = -2z$ and one of its generators having d.r.s. $1, 1, 1$.

20. Find the equation of the right circular cone generated by the lines drawn from the origin to cut the circle through the points $(1, 2, 2)$, $(2, 1, -2)$ and $(2, -2, 1)$.

21. Find the equation of the cone with vertex at the origin and containing the curve $ax^2 + by^2 = 2z; lx + my + nz = p$.

22. Obtain the equation of the right circular cone which is generated by revolving the line whose equations are $3x - y + z = 1; 5x + y + 3z + 1 = 0$ about the y-axis.

23. Find the equation of the cone which passes through the coordinate axes and has two generators having direction ratios 1,2,2 and -2,-2,1.

24. Obtain the equation of the cone which passes through the coordinate axes and has the lines $\dfrac{x}{2} = \dfrac{y}{1} = \dfrac{z}{3}$ and $\dfrac{x}{-3} = \dfrac{y}{1} = \dfrac{z}{-2}$ as its generators.

## 9.10 Answers

(1) $12x^2 - 4y^2 + z^2 + 4yz + 12zx + 4z - 4 = 0.$
(2) $25(x^2 + y^2) - 4z^2 = 0.$
(3) $12x^2 + 4y^2 + 3z^2 + 6yz + 8xy - 32x - 34y - 24z + 69 = 0.$
(4) $4x^2 + 4y^2 + 4z^2 + 9yz + 9zx + 9xy = 0.$
(5) $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0.$
(6) $(mx - ly)^2 + (mz - ny)^2 = m^2a^2.$
(7) $5x^2 + 8y^2 + 5z^2 - 4yz - 8zx - 4xy + 22x - 16y - 14z + 26 = 0.$
(8) $5x^2 + 5y^2 + 5z^2 - 8yz - 8zx - 8xy + 8x + 86y + 278 = 0.$
(10) $6x^2 + 6y^2 + 6z^2 - 13xz = 0.$



(11) $9(2y + z - 7)^2 + 4(z - 3x)^2 + (3x + 2y - 7)^2 = 196.$
(12) $3x^2 - 2y^2 + 5z^2 = 0.$
(13) $(b)$ and $(c)$ are generators;$(a)$ and $(d)$ are not generators.
(14) $x^2 + y^2 - z^2 = 0.$
(15) $2x^2 + y^2 - 3yz + 4zx - 5xy = 0.$
(16) $26x^2 + 29y^2 + 5z^2 + 10yz - 24zx - 4xy + 150y + 30z + 75 = 0.$
(17) $x^2 + y^2 + z^2 - zx + xy = 0.$ (18) $yz - zx - xy = 0.$
(20) $x^2 - 8y^2 - z^2 - 12yz + 6zx + 12xy - 46x + 38y + 22z - 19 = 0.$
(21) $8x^2 - 4y^2 - 4z^2 + yz + 5zx + 5xy = 0.$
(22) $apx^2 + bpy^2 - 2nz^2 - 2myz - 2lzx = 0.$
(23) $x^2 - 5y^2 + z^2 - 10y - 5 = 0.$
(24) $yz - zx - xy = 0.$ (25) $6yz - zx - 6xy = 0.$

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