CHAPTER 1

Real Numbers

1.1 Introduction

In this chapter, we shall study some basic properties of *Real Numbers.* We shall assume the familarity of the students with the sets of *Natural Numbers, Integers and Rational Numbers.* The set of real numbers can be constructed from the set of rational numbers or can also be defined axiomatically. However, the rigourous definition, in either of these two ways, is beyond the scope of this book. Here, we shall state the properties of real numbers that are used in the sequel. These properties can be classified in three catergories: *Alegebraic Properties, Order Properties and the property of 'Completeness'.* The algebraic and order properties have been used in high school mathematics. However, the *completeness property* is not studied before. The completeness property of real number system is most crucial in Calculus. In fact, the theory of calculus is founded on the completeness property of the real number system. Thus the emphasis of the present discussion is on the completeness property and its consequences. The students are advised to focus on the completeness property and urged to understand its fundametal role in the theory of calculus.

We now state the algebraic and order properties of real numbers.

(Algebraic Properties) : There are defined on $\mathbb{R}$ two binary operations, namely addition (+) and multiplication (.) which satisfy the following properties:

(i) Commutativity : For all $a, b \in \mathbb{R},\ a + b = b + a$ and $ab = ba$.
(ii) Associativity : For all $a, b, c \in \mathbb{R},\ a+(b+c) = (a+b)+c,\ a\cdot(b\cdot c) = (a\cdot b)\cdot c$.
(iii) There exist in $\mathbb{R}$, two elements 0 and 1 such that $0 \neq 1$ and for all $a \in \mathbb{R},\ a + 0 = a$ and $a \cdot 1 = a$.
(iv) For every $a \in \mathbb{R}$, there is an element $-a$ in $\mathbb{R}$ such that $a + (-a) = 0$.
(v) For every $a \in \mathbb{R}$ with $a \neq 0$, there is an element a^{-1} in $\mathbb{R}$ such that $a \cdot a^{-1} = 1$.
(vi) Distributive Law : For all $a, b, c \in \mathbb{R},\ a \cdot (b + c) = a \cdot b + a \cdot c$.

Note : We write ab in place of $a \cdot b$. The operations of subtraction and division are defined as follows: for $a, b \in \mathbb{R}$,

$$a - b = a + (-b) \quad \text{and if } a \neq 0, \text{ then } \quad \frac{a}{b} = ab^{-1}.$$

As $\mathbb{R}$ possesses the above algebraic properties, $\mathbb{R}$ is called a field.

(Order Properties) : There is defined on $\mathbb{R}$ a relation, called an order relation and denoted by $<$, which satisfies the following properties:
(vii) For all $a, b \in \mathbb{R}$, exactly one of the following statements is true:

$$a < b, \; a = b, \; b < a.$$

(viii) Transitivity: For all $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$ then $a < c$.
(ix) For all $a, b, c \in \mathbb{R}$,
 (α) $a < b \;\Rightarrow\; a + c < b + c$ and
 (β) $a < b$ and $0 < c \;\Rightarrow\; ac < bc$.
This last property states the relation between the algebraic operations and order.

We write $a < b < c$ to mean $[a < b$ and $b < c]$, $a \leq b$ to mean $[\, a < b$ or $a = b]$, and $b > a$ to mean $a < b$.

Statements such as $a < b, \; a \leq c$ are called *inequalities*. We say that a real number a is positive or negative according as $a > 0$ or $a < 0$.

The following are basic properties of inequalities:
1) $a > b \;\Leftrightarrow\; a - b > 0$.
2) $a > 0 \;\Leftrightarrow\; -a < 0$.
3) $[a > 0$ and $b > 0] \;\Rightarrow\; ab > 0$.
4) If $a < b$ then $ac < bc$ if c is positive and $ac > bc$ if c is negative.
This latter result says that multiplication by a *negative* number *reverses* the direction of inequality. For example,

$$2 < 5 \;\Rightarrow\; 2(-3) > 5(-3) \;\Rightarrow\; -6 > -15.$$

5) $a < b$ and $c < d \;\Rightarrow\; a = c < b + d$.
This says that, like equations, we may *add* inequalities (in the proper order).
6) $[0 \leq a < b$ and $0 \leq c < d] \;\Rightarrow\; ac < bd$.
This says that we may *multiply* inequalities when no negative numbers are involved. But this is false if negative numbers are involved. For example, it is true that $2 < 5$ and $-3 < -2$. But it is *not* true that $-6 < -10$.
7) $[0 \leq a < b$ and $0 < c < d] \;\Rightarrow\; a/d < b/c$.
8) $a \neq 0 \;\Rightarrow\; a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
9) $a > 0 \;\Rightarrow\; 1/a > 0$. Also, $a > b > 0 \;\Rightarrow\; 1/a < 1/b$.

10) $a > 1 \;\Rightarrow\; a^2 > a$ and $0 < a < 1 \;\Rightarrow\; a^2 < a$.
11) Let $b > 0,\; y > 0$. Then

$$\frac{a}{b} < \frac{x}{y} \;\Rightarrow\; \frac{a}{b} < \frac{a+x}{b+y} < \frac{x}{y}.$$

12) Let $a < b + \epsilon$ for *every positive* number ϵ. Then $a \leq b$.
This result is often useful. To prove it, let $a > b$, if possible. Then $a - b > 0$. Hence taking $\epsilon = a - b$ we get $a < b + (a - b)$ or $a < a$. This is a contradiction. Hence $a \leq b$.

Definition: Suppose S is a non-empty subset of $\mathbb{R}$. A real number a is called a *minimum* element of S and we write $a = \min S$ if (i) $a \in S$ and (ii) $a \leq x$ for all x in S.

A real number b is called a *maximum* element of S and we write $b = \max S$ if (i) $b \in S$ and (ii) $x \leq b$ for all x in S.

A minimum element of S is unique (if it exists). To see this, let a, a' be both minimum elements of S. Then by definition, we must have $a \leq a'$and $a' \leq a$. Hence $a = a'$. Similarly, a maximum element of S is unique (if it exists).

1.3 Rational Numbers

We denote by $\mathbb{N},\; \mathbb{Z}$ and $\mathbb{Q}$ the set of all Natural numbers, Integers and Rational numbers respectively.

The following property of $\mathbb{N}$ is equivalent to the Principle of Mathematical Induction:
Well-odering principle : Every non-empty subset of $\mathbb{N}$ contains a minimum element.

Also note that
(i) $1 = \min \mathbb{N}$ and if $n \in \mathbb{N}$ and $x \in \mathbb{R}$ are such that $n < x < n+1$, then $x \notin \mathbb{N}$.
(ii) The set of all rational numbers is an ordered field (like $\mathbb{R}$).
(iii) $\mathbb{Q}$ is a *dense* set. That is, between any two distinct rational numbers there is a rational number.

For, if $a, b \in \mathbb{Q}$, and $a < b$, then $c = (a + b)/2 \in \mathbb{Q}$ and $a < c < b$.

1.5 Absolute Value

Definition: For every $x \in \mathbb{R}$, the absolute value of x is denoted by $|x|$ and is defined as follows:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

In other words, $|x| = \max\{x, -x\}$.

For example, $|2| = 2,\ |-3| = 3,\ |0| = 0$.

Also, if real numbers a, b correspond to points A, B on the real line, then we define the number $|a - b|$ to be the (non-negative) *distance* between A and B.

Let x, y, a, b be any real numbers. We now list properties of absolute value. Most of them can be proved by considering *cases*. The results 8,9,10 are particularly important.

1. $|x| \geq 0$. Also, $|x| = 0$ if and only if $x = 0$.

2. $|-x| = |x|$. Hence $|x - y| = |y - x|$.

3. $|xy| = |x||y|$.

4. $|x^2| = |x|^2 = x^2$.

   Hence $|x|$ is the non-negative square-root of x^2.

5. $|x/y| = |x|/|y|$, if $y \neq 0$.

6. Let $a \geq 0$. Then $|x| \leq a$ if and only if $-a \leq x \leq a$.

   To prove $|x| \leq a \Rightarrow -a \leq x \leq a$, let $|x| \leq a$. Then

   $x \geq 0 \Rightarrow [-x \leq 0 \leq a$ and $x = |x| \leq a]$ so that $-a \leq x \leq a$.

   $x < 0 \Rightarrow x < 0 \leq a$ and $-x = |x| \leq a$ and so $-a \leq x \leq a$ again.

   The converse can be proved similarly.

7. $-|x| \leq x \leq |x|$.

8. $|x \pm y| \leq |x| + |y|$.

   For, by 7, $-|x| \leq x \leq |x|,\ -|y| \leq y \leq |y|$.

   Hence adding we get $-(|x| + |y|) \leq x + y \leq |x| + |y|$.

   So by 6, $|x + y| \leq |x| + |y|$.

Alternatively, it is easy to see that $|x+y| = |x| + |y|$ if x, y have the same sign and $|x+y| < |x| + |y|$ if x and y are of opposite signs.

Replacing y by $-y$ in the last result we get

$|x-y| \leq |x| + |-y| = |x| + |y|$.

9. $||x| - |y|| \leq |x \pm y|$.

   For, $\quad |x| = |(x-y)+y| \leq |x+y| + |y|$.

   Hence $\quad |x| - |y| \leq |x+y|$.

   Also, $\quad -(|x| - |y|) = |y| - |x| \leq |y+x| = |x+y|$.

   So by 6, $||x| - |y|| \leq |x+y|$. Finally, replacing y by $-y$,

   $$|x-y| \geq ||x| - |-y|| = ||x| - |y||.$$

10. Let $|a| \neq |b|$. Then
    $$\left|\frac{x+y}{a+b}\right| \leq \frac{|x|+|y|}{|a|-|b|}.$$

    For by 8 and 9,

    $0 \leq |x+y| \leq |x| + |y| \quad \text{and} \quad 0 < ||a| - |b|| \leq |a+b|$.

    Hence by 5 above and 7 of §1.2 we get

    $$\left|\frac{x+y}{a+b}\right| = \frac{|x+y|}{|a+b|} \leq \frac{|x|+|y|}{|a|-|b|}.$$

11. Let $a > 0$. Then $|x-y| < a \;\Leftrightarrow\; y-a < x < y+a$.

12. Let $a > 0$. Then $|x| \leq a \;\Leftrightarrow\; x^2 \leq a^2$

    and $|x| > a \;\Leftrightarrow\; x < -a$ or $x > a$.

13. If $a_1, a_2, \ldots, a_n$ are any finitely many real numbers, then

    $$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

Notation: Given $a, b \in \mathbb{R},\ a < b$, we define the following sets:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

$$[a,b) = \{x \in \mathbb{R} \mid a \le x < b\},$$
$$(a,b] = \{x \in \mathbb{R} \mid a < x \le b\}.$$

The set $[a,b]$ is called a *closed interval* with *end points a,b* and *length* $b-a$. The set (a,b) is called an *open interval* and $[a,b)$ and $(a,b]$ are called *half open/half closed intervals.*

In calculus many times we need to consider a set of those points which are *near* a given point. Such a set is called a *neighbourhood* of the given point. Thus given a number a and a positive number δ, we write

$$N(a,\delta) = (a-\delta, a+\delta)$$

and call this open interval the δ-neighbourhood (briefly, δ-nhd) of a. In other words, $N(a,\delta)$ is the set of those points which are distance *less than* δ from a :

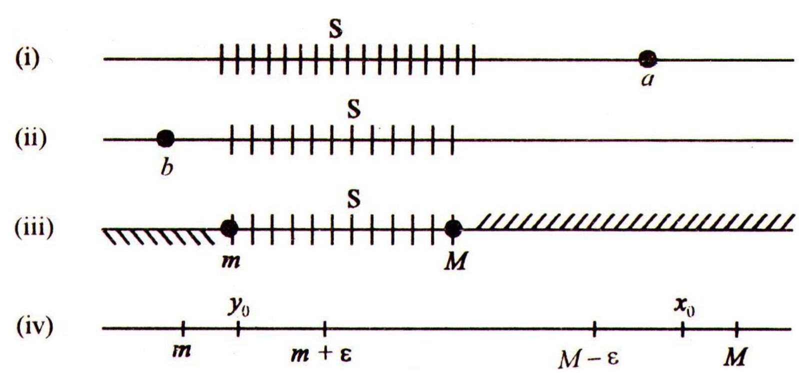
$$N(a,\delta) = \{x \mid |x-a| < \delta\}.$$

Also, omitting the point a from $N(a,\delta)$ we obtain the *deleted* δ-nhd of a, which is denoted as $N'(a,\delta)$. Thus

$$\begin{aligned} N'(a,\delta) &= N(a,\delta) - \{a\} \\ &= (a-\delta,a) \cup (a,a+\delta) \\ &= \{x \mid 0 < |x-a| < \delta\}. \end{aligned}$$

1.6 Supremum and Infimum

Definition : Let S be a non-empty set of real numbers. Then



(i) S is said to be *bounded above* if there is a real number a such that $x \leq a$ for all $x \in S$. Also, then a is called a *rough upper bound* or just an *upper bound* for S. See Fig.1(i).

(ii) S is said to be *bounded below* if there is a real number b such that $b \leq x$ for all $x \in S$. Also, then b is called a *rough lower bound* or just a *lower bound* for S. See Fig.1(ii).

(iii) S is said to be *bounded* if S is bounded above and also bounded below. See Fig.1(iii).

Examples : 1) The set $S = \{1, 3, 4, 6\}$ is bounded above and for example, 8 is a rough upper bound for S since *all* elements of S are less than 8. Similarly, $6, 8.52, 15\sqrt{7}$ are also upper bounds for S. In fact, *every number* ≥ 6 is an upper bound for S. Also, S is bounded below and *every number* $le1$ is a lower bound for S. Hence by (iii) of the above definition the set S is a bounded set.
2) Consider the interval $T = [3, 7.4)$. By definition, we know that T is the set of all numbers x such that $3 \leq x < 7.4$. Hence T is bounded above and every number ≥ 7.4 is an upper bound for T. Also T is bounded below and every number ≤ 3 is a lower bound for T. Hence T is a bounded set.
3) Let S be a non-empty set of real numbers. Then

$$\begin{aligned} & S \text{ is a bounded set} \\ \Leftrightarrow \quad & \text{There are real numbers } a, b \text{ such that } a \leq x \leq b \text{ for all } x \in S \\ \Leftrightarrow \quad & S \subseteq [a, b] \text{ for some real numbers } a, b. \end{aligned}$$

4) The set $\mathbb{R}$ is *not* bounded above. To prove thissuppose that $\mathbb{R}$ is bounded above and let a be an upper bound for $\mathbb{R}$. Then *every* number in $\mathbb{R}$ must be less than or equal to a. But this is not true. for example, $a + 1$ is an element of $\mathbb{R}$ and $a + 1 > a$. This contradiction proves that $\mathbb{R}$ is not bounded above. Similarly, it can be shown that $\mathbb{R}$ is not bounded below.

Consider the above set $S = \{1, 3, 4, 6\}$ again. Clearly, 6 is an upper bound for S. But the number 6 has the following *additional* property: No number smaller than 6 is an upper bound for S. To see this let a be *any* number < 6. Then it is not true that every element of S is $\leq a$. For example, 6 is in S and $6 > a$. Hence a is not an upper bound for S. The above two properties of 6, taken together, mean that 6 is the *smallest* among all the upper bounds for S. We say that 6 is the *least* upper bound for S. Similarly, 1 is a lower bound for

S and moreover, no number greater than 1 is a lower bound for S. Thus 1 is the *greatest* lower bound for S.

Definition : Let S be a no-empty set of real numbers. A real number M is called the *least upper bound* or the *supremum* for S, and we write $M = \sup S$, if

(i) M is an upper bound for S and
(ii) no number less than M is an upper bound for S. See Fig.1(iii).

Note that property (ii) can be stated equivalently as follows:

(ii)$'$ For *every* number $\epsilon > 0$, the number $M - \epsilon$ is *not* an upper bound for S. That is,
(ii)$''$ For *every* number $\epsilon > 0$, there is at least one element x_0 of S such that $x_0 > M - \epsilon$. See Fig.1(iv).

Definition : Let S be a no-empty set of real numbers. A real number m is called the *greatest lower bound* or the *infimum* for S, and we write $m = \inf S$, if

(i) m is a lower bound for S and
(ii) no number greater m is a lower bound for S. See Fig.1(iii).

Note that property (ii) can be stated equivalently as follows:

(ii)$'$ For *every* number $\epsilon > 0$, the number $m + \epsilon$ is *not* a lower bound for S. That is,
(ii)$''$ For *every* number $\epsilon > 0$, there is at least one element y_0 of S such that $y_0 < m + \epsilon$. See Fig.1(iv).

The numbers $\inf S$ and *supS* are also sometimes called the *exact bounds* of S.

Examples : 5) For the set $S = \{1, 3, 4, 6\}$, we have $\sup S = 6$ and $\inf S = 1$.
6) For $T = [3.7.4)$, $\sup T = 7.4$ and $\inf T = 3$.

To prove this observe that 7.4 is an upper bound for T as $x < 7.4$ for all $x \in T$. Secondly, take any number $a < 7.4$. Then if $a < 3$, take $x_0 = 3$, and if $3 \leq a < 7.4$, take $x_0 = (a + 7.4)/2$. In either case we see that $x_0 \in T$ and $x_0 > a$. Hence a is not an upper bound for T and so $\sup T = 7.4$.

Next, 3 is clearly a lower bound for T. Also if c is any number greater than 3, then $y_0 = 3$ is an element of T such that $y_0 < c$. So c is not a lower bound for T. Hence $\inf T = 3$.

7) Let S be a no-empty set of real numbers. Then the supremum of S is unique, if it exists. Similarly, $\inf S$ is unique, if it exists.

To prove this, suppose that S has two suprema, say M and M'. If these are unequal then either $M < M'$ or $M > M'$. First suppose $M < M'$. Then since M' is a supremum of S, M is not an upper bound of S. This contradicts our assuption that M is a supremum of S. We get a similar contradition if $M > M'$. Hence $M = M'$ and so S can have at most one supremum. Similarly, it can be shown that $\inf S$ is unique, when it exists.
8) The supremum of a set $S \subseteq \mathbb{R}$, if it exists, may or may not be an element of S. A similar remark is true for the infimum of S.

Thus in Ex. 6 above, $\sup T = 7.4$ is *not* in $T = [3, 7.4)$ while in Ex. 5, $\sup S = 6$ actually *is* an element of $S = \{1, 3, 4, 6\}$. Similarly, it is easy to see that $\inf T = 3$ and $3 \in T$ while $\inf(4, 9] = 4$ and 4 is not in the half-open interval $(4, 9]$.

But if $\sup S$ exists and *belongs to* S, then $\sup S$ is clearly the maximum element of S.

Conversely, if $\max S$ exists then $\sup S$ exists and $\max S = \sup S$. Thus in Ex. 5, $\sup S = \max S = 6$. But the set T in Ex. 6 has no maximum element because $\sup T$ exists but is not in T.

Similarly, $\inf S$ exists and belongs to S if and only if $\min S$ exists; and in either case, $\inf S = \min S$. Thus $\inf[3, 7.4) = \min[3, 7.4) = 3$.

Finally, the open interval $A = (4, 9)$ has no minimum element because $\inf A$ exists and is 4 but $4 \notin A$.
9) Suppose S is a non-empty *finite* set of real numbers. Then by the property 6 of §1.4, S possesses the maximum element and the minimum element. Hence in this case, $\sup S$ and $\inf S$ both exist and are respectively equal to $\max S$ and $\min S$.

This is not true for infinite sets of real numbers: for example, the infinite set $\mathbb{N}$ has no maximum element.

Therefore the distinction between $\sup S$ and $\max S$ (and also between $\inf S$ and $\min S$) appears *only when* we consider *infinite* sets.
10) By Ex. 8 and 9, we see that $T = [3, 7.4)$ is a bounded infinite set.

Note: Clearly, if $\inf S$ and $\sup S$ both exist, then we have

$$\inf S \leq x \leq \sup S, \quad \text{for all } x \in S.$$

We now state the *Completeness Property* of real numbers.

The Completeness Property: For every non-empty subset S of $\mathbf{R}$, if S is bounded above, then $\sup S$ exists in $\mathbb{R}$. (This is expressed by saying that the ordered field $\mathbb{R}$ is *complete*.)

This property gaurantees the *existence* of the supremum of a non-empty subset of $\mathbb{R}$ which is bounded above. From it a similar result for infimum can be deduced; see property 2 below.

The ordered field $\mathbb{Q}$ of rational numbers is not complete in the above sense. To prove this consider the set

$$A = \{x \in \mathbb{Q} \mid x^2 < 2\}.$$

Then A is non-empty because for example, $1 \in S$. Next, 2 is an upper bound for A because $2 \in \mathbb{Q}$ and

$$x \geq 2 \;\Rightarrow\; x^2 \geq 4 > 2 \;\Rightarrow\; x^2 > 2 \;\Rightarrow\; x \notin A,$$

so that $x < 2$ for all x in A.

Hence A is bounded above. But $\sup A$ *does not exist in* $\mathbb{Q}$. To see this take any rational number a. Then by 5 of §1.3,$a^2 \neq 2$. Hence either $a^2 < 2$ or $a^2 > 2$.

If $a \leq 0$, then $a < 1$ and $1 \in A$. Hence a is not an upper bound of A.

Let $a > 0$ and $a^2 < 2$. Consider the number $b = (2a+2)/(a+2)$. Then $b \in \mathbf{Q}, b > 0$ and

$$\text{(i)}\;\; b^2 - 2 = \frac{2(a^2-2)}{(a+2)^2} \qquad \text{(ii)}\;\; b - a = \frac{2-a^2}{a+2}.$$

By (i), $b^2 < 2$. Hence $b \in A$. Also by (ii), $b > a$. Hence a is not an upper bound of A.

Finally, let $a > 0$ and $a^2 > 2$. Then a is an upper bound of A. (Why ?). But taking b as before we see that $b \in \mathbb{Q},\; b > 0$ and by (i), $b^2 > 2$. Hence b is an upper bound of A. Also, by (ii), $b < a$. Hence b is an upper bound of A and b is *smaller* than a. Hence a is not the smallest upper bound for A, i.e. $a \neq \sup A$.

Thus we see that the subset A of $\mathbb{Q}$ is non-empty and is bounded above but $\sup A$ does not exist *in* $\mathbb{Q}$.

The following properties of real numbers can be deduced from the completeness property.

1) Existence of *irrational* numbers : Consider the set

$$B = \{x \in \mathbb{R} \mid x^2 < 2.\}.$$

Then, as seen for the set A above, we can show that B is a non-empty subset of $\mathbb{R}$ and B is bounded above. Hence by completeness property, $\sup B$ $(= a$, say) exists in $\mathbb{R}$. Clearly, $a > 0$. Also, as seen above,

$$a^2 < 2 \;\Rightarrow\; a \neq \sup B \;\text{ and }\; a^2 > 2 \;\Rightarrow\; a \neq \sup B.$$

Hence $a^2 = 2$. This number a is a real irrational number. It is clled the positive square root of 2 and is denoted by $\sqrt{2}$.

Note : Suppose $a, b \in \mathbb{R}$ and a is irrational and b is rational. Then the numbers $a + b, -a, a - b$ and a^{-1} are irrational. If $b \neq 0$, then the numbers ab, ab^{-1}, ba^{-1} are irrational.

2) Let S be a non-empty set of real numbers. If S is bounded below, then $\inf S$ exists in $\mathbb{R}$.

To prove this, consider the set

$$S_1 = \{-x \mid x \in S\}.$$

Then S_1 is non-empty as S is non-empty. Also,

$$\begin{aligned} a \text{ is a lower bound for } S \;&\Leftrightarrow\; a \leq x, \text{ for all } x \in S \\ &\Leftrightarrow\; -a \geq -x, \text{ for all } x \in S \\ &\Leftrightarrow\; -a \geq y, \text{ for all } y \in S_1 \\ &\Leftrightarrow\; -a \text{ is an upper bound for } S_1. \end{aligned}$$

Now suppose that S is bounded below. Then by the above, S_1 is bounded above and so, by completeness property , $\sup S_1(= M$, say) exists in $\mathbb{R}$. Then again by the above, $-M$ is a lower bound for S. Also, given $\epsilon > 0$, there is $x_0 \in S_1$ such that $x_0 > M - \epsilon$. Hence $y_0 = -x_0 < -M + \epsilon$ and $y_0 \in S$. Hence $-M = \inf S$.

3) **Archimedean property :** If $x, y \in \mathbb{R}$, then there is a natural number n such that $nx > y$.

To prove this, note first that if $x \geq y$ then the result holds with $n = 1$. Next, let $0 < x < y$. Suppose that there is no natural number n such that

$nx > y$. Then for all $n \in \mathbb{N}$, we have $nx < y$. Hence the set $S = \{nx \mid n \in \mathbb{N}\}$ is non-empty and bounded above. Hence by completeness property $\sup S (= M$, say) exists. Then

$$\begin{aligned} \text{For all } n \in \mathbb{N} \qquad & nx \leq M \\ \Rightarrow \quad & \text{For all } n \in \mathbb{N},\ (n+1)x \leq M \\ \Rightarrow \quad & \text{For all } n \in \mathbb{N},\ nx \leq M - x. \end{aligned}$$

Now $M - x < M$ as $x > 0$. So $M - x$ is an upper bound of S and $M - x < \sup S$. This is a contradiction. Hence $nx \geq y$ for some $n \in \mathbb{N}$.

4) For every $y \in \mathbb{R}$, then there is a natural number n such that $n > y$.

With $x = 1$, the Archimedean property implies that there is a natural number m such that $m = mx > y$. So $n = m + 1$ is as required.

5) If $x \in \mathbb{R}$ and $x > 0$, then there is a natural number n such that $0 < 1/n > x$.

Here if $x > 1$, take $n = 1$. If $0 < x \leq 1$, the Archimedean property implies that there is a natural number m such that $mx \geq 1$. Now let $n = m + 1$. Then $nx > mx \geq 1$ and so $nx > 1$ as required.

6) Given $x \in \mathbb{R}$, there is a unique integer m such that $m \leq x < m + 1$. This integer is called the *integral part of* x and is denoted as $[x]$.

To prove uniqueness of $[x]$, let m, n be two integers such that

$$m \leq x < m + 1 \quad \text{and} \quad n \leq x < n + 1.$$

Then $m < n + 1$ and $n < m + 1$. hence $m \leq n$ and $n \leq m$; so $m = n$.

To prove the existence of $[x]$, we consider various cases:

(i) Observe that $[x] = 0$ if $0 \leq x < 1$ and $[x] = -1$ if $-1 \leq x < 0$.

(ii) If $x \geq 1$, by 4) there is a natural number such that $n > x$. of all such natural numbers n, let $m + 1$ be the *least*, which exists by the well-ordering principle. Then $m \leq x$. Thus $m \leq x < m + 1$ and so $[x] = m$.

(iii) If $x < -1$, then $-x > 1$ and so by 4) there is a natural number such that $n \geq x$. of all such natural numbers n, let $p + 1$ be the *least*, which exists by the well-ordering principle. Then $p < -x \leq p + 1$ so that $-p - 1 \leq x < p$ and so $[x] = -p - 1$.

7) The set $\mathbb{Q}$ is dense in $\mathbb{R}$. That is, between any two distinct real numbers there is a rational number.

Suppose $a, b \in \mathbb{R}$ and $a < b$. Then $b - a > 0$ and so by 5), there is a natural number n such that $1/n < b - a$. By 6), there is an integer m such that

$$\begin{array}{rrcl} & m-1 & \leq & an+m \\ \text{or} & an & < & m \leq an+1 \\ \text{or} & a & < & \dfrac{m}{n} \leq a + \dfrac{1}{n}. \end{array}$$

Thus m/n is a rational number between a and b.

8) The set all irrational numbers is dense in $\mathbb{R}$. That is, between any two distinct real numbers there is an irrational number.

Suppose $a, b \in \mathbb{R}$ and $a < b$. If $a > 0$ or $b < 0$, by 7), choose a rational number r such that $a\sqrt{2} < r < b\sqrt{2}$. If $a < 0 < b$, again by 7) choose a rational number r such that $0 < r < b\sqrt{2}$. Then as $\sqrt{2} > 0$, we get on dividing by $\sqrt{2}$, $a < r/\sqrt{2} < b$ and $c = r/\sqrt{2}$ is an irrational number as r is a non-zero rational and $\sqrt{2}$ is irrational. Thus c is an irrational number between a and b.

Examples : 1) Consider the set

$$S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}.$$

Here every element x of S is of the form $x = \dfrac{1}{n}$ for some $n \in \mathbb{N}$. Now 1 is the maximum element of S because $1 \in S$ and $1/n \leq 1$ for all $n \in \mathbb{N}$. Hence $\sup S = 1$. Next, $\inf S = 0$. For, first $\dfrac{1}{n} > 0$. Secondly, given any $\epsilon > 0$, by 5), there is a natural number m such that $\dfrac{1}{m} < \epsilon$. Hence if $x_0 = \dfrac{1}{m}$, then $x_0 \in S$ and $x_0 < 0 + \epsilon$. So $\inf S = 0$. Finally, S has no minimum element because $\inf S = 0 \notin S$.

2) Let S, T be non-empty sets of real numbers and $S \subseteq T$. Then

(i) $\sup S \leq \sup T$ if T is bounded above.

(ii) $\inf T \leq \inf S$ if T is bounded below.

To prove (i), suppose that T is bounded above. Then there is a number k such that $x \leq k$ for all $x \in T$. In particular, $x \leq k$ for all $x \in S$ since $S \subseteq T$. Hence S is also bounded above. So by the completeness property, $\sup T$ and $\sup S$ both exist. Now $x \in S \Rightarrow x \in T \Rightarrow x \leq \sup T$. Hence $\sup T$ is an upper bound of S. Therefore $\sup S \leq \sup T$ since $\sup S$ is the *least* upper bound of S. This proves (i). The reader can similarly prove (ii).

EXERCISE 1

1. Solve the inequalities:

   (i) $3 - 5x < 2x - 11$ (ii) $-1 < \dfrac{3x+4}{x-7}$.

2. Let $a > 0$ and $f(x) = ax^2 + bx + c$. Prove that

   (i) $f(x) \geq 0$ for all $x \in \mathbb{R}$, $\Leftrightarrow$ $b^2 \geq ac$.

   (ii) If $f(x) = 0$ has distinct real roots α, β and $\alpha < \beta$, then for a given number t,
   $$f(t) > 0 \ \Leftrightarrow\ t < \alpha \ \text{ or } \ x > \beta,$$
   $$f(t) < 0 \ \Leftrightarrow\ \alpha < t < \beta.$$

3. Solve the inequalities:

   (i) $|3x+4| < |x+2|$ (ii) $\dfrac{2+x}{3-x}$ (iii) $|2x^2 - 11x + 14| < 2$.

4. Prove that a set of real numbrs is bounded if and only if there is a constant $a > 0$ such that $|x| \leq a$ for all $x \in S$.

5. Find the supremum and infimum of the following sets:

   (i) (a, b) (ii) (a, ∞) (iii) $(-\infty, a)$ (iv) $\{(-1)^n \dfrac{1}{n} : n \in \mathbb{N}\}$

   (v) $\{\dfrac{n-1}{n} : n \in \mathbb{N}\}$ (vi)$\{(-1)^n + \dfrac{n+1}{n+2} : n \in \mathbb{N}\}$

6. If $a, b \in \mathbb{R}$, show that

   (i) $\max\{a, b\} = \dfrac{1}{2}\Big[a + b + |a - b|\Big]$,

   (ii) $\min\{a, b\} = \dfrac{1}{2}\Big[a + b - |a - b|\Big]$.

7. Find a rational number between $\sqrt{6}$ and $\sqrt{7}$.

ANSWERS

1. (i) $x > 2$ (ii) $x < \dfrac{3}{4}$ or $x > 7$.

2. (i) Use the identity $f(x) = a\Big[\Big(x + \dfrac{b}{a}\Big)^2 + \dfrac{ac - b^2}{a^2}\Big)\Big]$.

   (ii) Here $f(x) = a(x - \alpha)(x - \beta)$.

3. (i) By §1.5 12), the inequality is equivalent to

$$(3x + 4)^2 < (x + 2)^2$$

$$\text{i.e. to} \qquad (2x + 3)(x + 1) < 0 \quad \text{or} \quad -\frac{3}{2} < x < -1.$$

   (ii) $x < \dfrac{1}{2}$ (iii) $\dfrac{3}{2} < x < 4$.

4. Suppose that the number a exists as stated. Then for all $x \in S$, we have $-a \leq x \leq a$. Hence S is bounded. Conversely, suppose S is bounded. Then there exist constants c, d such that $c \leq x \leq d$ for all $x \in S$. Let $a \geq \max\{-c, d\}$. Then for all $x \in S$,

$$-a \leq c \leq x \leq d \leq a \text{ or } |x| \leq a.$$

5. (i) $\sup = b, \inf = a$ (ii) sup does not exist, $\inf = a$
   (iii) $\sup = a$, inf does not exist (iv) $\sup = \frac{1}{2}, \inf = -1$
   (v) $\sup = 1, \inf = 0$ (vi) $\sup = 2, \inf = \frac{-1}{3}$

6. Since $600 < 625 < 700$, we have

$$10\sqrt{6} < 25 < 10\sqrt{7}$$

$$\text{or} \qquad \sqrt{6} < 2.5 < \sqrt{7}.$$

   So we may take $r = 2.5$.

CHAPTER 2

Limits

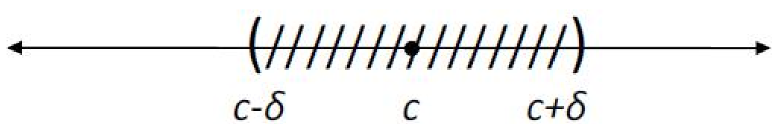
Introduction :

Limit is the fundamental concept of Calculus. It describes the behaviour of a function as its input value approaches a particular value. We will introduce the concept of limit in this chapter and study its basic properties.

Suppose a man wants to compute the speed of a vehicle passing through a point P. He can determine the instantaneous speed by computing the average speed from point P to points which are *close* to P. If these average speeds over small distances approach a certain value, then that value is known as the instantaneous speed at P. This is exactly how the speed of a vehicle is determined in real-world models.

Neighbourhood of a point: Let $c \in \mathbb{R}$ and $\delta > 0$ be any positive real number, then the neighbourhood of the point c of radius δ is denoted by $N_\delta(c)$ and is defined as

$$\begin{aligned} N_\delta(c) &= \{x \in \mathbb{R} \mid |x - c| < \delta\} = \{x \in \mathbb{R} \mid -\delta < x - c < \delta\} \\ &= \{x \in \mathbb{R} \mid c - \delta < x < c + \delta\} = (c - \delta, c + \delta) \\ &= \text{the open interval from } \ c - \delta \ \text{ to } \ c + \delta. \end{aligned}$$



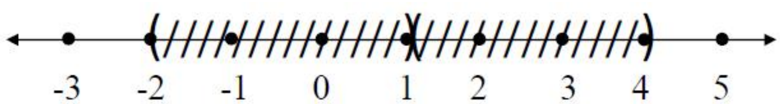
Example : $N_{0\cdot 2}(3) = (3 - 0 \cdot 2, 3 + 0 \cdot 2) = (2 \cdot 8, 3 \cdot 2)$.

Example : $N_5(2) = (2 - 5, 2 + 5) = (-3, 7)$.

Deleted Neighbourhood of a point: Let $c \in \mathbb{R}$ and $\delta > 0$ be any positive real number, then the deleted neighbourhood of the point c of radius δ is denoted by $N'_\delta(c)$ and is defined as

$$\begin{aligned} N'_\delta(c) &= \{x \in \mathbb{R} \mid 0 < |x-c| < \delta\} \\ &= \{x \in \mathbb{R} \mid -\delta < x-c < \delta\} - \{c\} \\ &= \{x \in \mathbb{R} \mid -\delta < x-c < \delta,\ x \neq c\} \\ &= \{x \in \mathbb{R} \mid c-\delta < x < c+\delta,\ x \neq c\} \\ &= (c-\delta, c+\delta) - \{c\} = (c-\delta, c) \cup (c, c+\delta). \end{aligned}$$

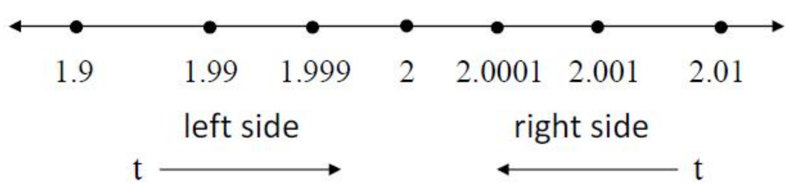
Example : $N'_3(1) = (1-3, 1) \cup (1, 1+3) = (-2,1) \cup (1,4)$.



Example : Suppose a particle falls freely experiencing no air resistance, and its velocity $v(t)$ at time t is given by $v(t) = 32t$. Consider the following table of values of $v(t)$:

t	$1 \cdot 9$	$1 \cdot 99$	$1 \cdot 999$	$2 \cdot 0001$	$2 \cdot 001$	$2 \cdot 01$
$v(t)$	$60 \cdot 8$	$63 \cdot 68$	$63 \cdot 968$	$64 \cdot 0032$	$64 \cdot 032$	$64 \cdot 32$

From the bottom row of the above table we see that the velocity $v(t)$ seems to be approaching the value 64 as the time t approaches 2. Observe that t can approach 2 from the right side of 2 or from its left side; but $v(t)$ approaches 64 in both cases.

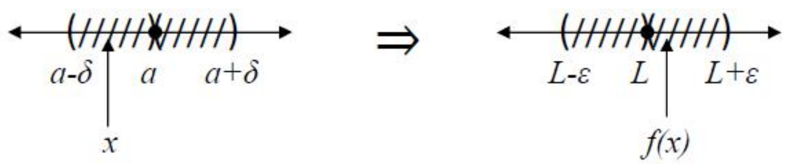


We express the above observation mathematically as

$$\lim_{t \to 2} v(t) = 64,$$

and say that "the limit of $v(t)$ as t approaches 2 is 64".

Formal Definition of Limit: Let f be a function defined in a deleted neighbourhood of a point $a \in \mathbb{R}$. We say that $L \in \mathbb{R}$ is the limit of the function f at a and write $\lim_{x\to a} f(x) = L$ if
for any $\epsilon > 0, \exists$ a $\delta > 0$ such that
  if $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$
i.e. in terms of neighbourhoods,
  if $x \in N'_\delta(a)$ then $f(x) \in N_\epsilon(L)$.



Note: We sometimes use the following notation: '$\exists$' is read as 'there exist' or 'there exists'. Also, '$\forall$' is read as 'for every' or 'for all'.

Example: Let $f(x) = b$, $\forall x \in \mathbb{R}$, be a constant function. Let $a \in \mathbb{R}$. Show that $\lim_{x\to a} f(x) = b$.

Solution: Let $\epsilon > 0$ be given, we have to find $\delta > 0$, such that,
  if $0 < |x-a| < \delta$ then $|f(x) - b| < \epsilon$.
Note that in this case, $|f(x) - b| = |b - b| = 0 < \epsilon \quad \forall x \in \mathbb{R}$. Hence we can take δ to be *any* positive number and then we can say that
  $0 < |x-a| < \delta \Rightarrow |f(x) - b| < \epsilon$.
  $\therefore \lim_{x\to a} f(x) = b$ i.e. $\lim_{x\to a} b = b$.

Example: Let $f(x) = x$, $\forall x \in \mathbb{R}$, be the identity function. Let $a \in \mathbb{R}$. Show that $\lim_{x\to a} f(x) = a$.

Solution: Let $\epsilon > 0$ be given, we have to find $\delta > 0$, such that,
  if $0 < |x-a| < \delta$ then $|f(x) - a| < \epsilon$.
Note that in this case, $|f(x)-a| = |x-a|$. So we take $\delta = \epsilon$. Then $0 < |x-a| < \delta \Rightarrow |f(x) - a| < \epsilon$.
$\therefore \lim_{x\to a} f(x) = a$ i.e. $\lim_{x\to a} x = a$.

Example: Using the definition of limit, show that $\lim_{x\to a} x^2 = a^2$ for any $a \in \mathbb{R}$.

Solution: Let $\epsilon > 0$ be given, we have to find $\delta > 0$, such that
  $0 < |x-a| < \delta \Rightarrow |x^2 - a^2| < \epsilon$.

Note that

$$\begin{aligned}
|x^2 - a^2| &= |(x-a)(x+a)| \\
&= |x-a|\,|x+a| \\
&\leq |x-a|(|x|+|a|) \qquad \text{[by triangle inequality]} \\
&< |x-a|(|a|+1+|a|) \qquad \forall\, x \in N_1(a) \\
&= |x-a|(2|a|+1) \qquad \forall\, x \in N_1(a) \\
&< \epsilon, \\
\text{if} \quad |x-a| &< \frac{\epsilon}{2|a|+1}.
\end{aligned}$$

So we choose $\delta = \min\left\{\frac{\epsilon}{2|a|+1}, 1\right\}$. Then by the above,

$0 < |x-a| < \delta \Rightarrow |x^2 - a^2| < \epsilon$.

$\therefore \lim_{x\to a} x^2 = a^2$.

Example: Using the defintion of limit show that $\displaystyle\lim_{x\to 0} \frac{2x^2+3}{x+5} = \frac{3}{5}$.

Solution: Let $\epsilon > 0$ be given, we have to find $\delta > 0$, such that

$0 < |x-0| < \delta \Rightarrow \left|\frac{2x^2+3}{x+5} - \frac{3}{5}\right| < \epsilon$.

Note that

$$\begin{aligned}
\left|\frac{2x^2+3}{x+5} - \frac{3}{5}\right| &= \left|\frac{10x^2+15-3x-15}{5(x+5)}\right| = \left|\frac{10x^2-3x}{5(x+5)}\right| \\
&= \left|\frac{x(10x-3)}{5(x+5)}\right| = |x|\frac{|10x-3|}{5|x+5|} \\
&\leq |x|\frac{|10x+50|}{5|x+5|} \qquad \forall\, x \in N_1(0) \\
&= |x|\frac{10|x+5|}{5|x+5|} \qquad \forall\, x \in N_1(0) \\
&= 2|x-0| \qquad \forall\, x \in N_1(0) \\
&< \epsilon,
\end{aligned}$$

if $|x-0| < \epsilon/2$. Hence choose $\delta = \min\left\{\frac{\epsilon}{2}, 1\right\}$. Then

$0 < |x-0| < \delta \Rightarrow \left|\frac{2x^2+3}{x+5} - \frac{3}{5}\right| < \epsilon$.

$\therefore \displaystyle\lim_{x\to 0} \frac{2x^2+3}{x+5} = \frac{3}{5}$.

Exercises:

Using the definition of limit show that

a) $\lim_{x\to 2} x^2 = 4$ b) $\lim_{x\to 3}(x^2 - x) = 6$ c) $\lim_{x\to 9}\sqrt{x} = 3$

d) $\lim_{x\to -6}\dfrac{x+4}{2-x} = \dfrac{-1}{4}$ e) $\lim_{x\to 3}\dfrac{x}{4x-9} = 1$ f) $\lim_{x\to 0}\sqrt{x} = 0.$

Non-existence of limit: Sometimes it so happens that a given function f does *not* have a limit at a given point a. That is, there exists *no number* L for which the definition of limit is satisfied. But how can we *prove* this fact in a given case?

For this purpose we first derive a *necessary* condition for the existence of the limit of $f(x)$ as $x \to a$.

Thus suppose that $\lim_{x\to a} f(x) = L$ for a certain number L. Then, by definition, given any $\epsilon > 0$, there exists $\delta > 0$ such that for all x the deleted neighbourhood $(a-\delta, a+\delta) - \{a\}$, we have

$$|f(x) - L| < \epsilon/2. \tag{A}$$

Hence, if x_1, x_2 are *any two* points in $(a-\delta, a+\delta) - \{a\}$, we have

$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - L + L - f(x_2)| \\ &\leq |f(x_1) - L| + |f(x_2) - L| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned} \tag{B}$$

In other words, condition (A) says that the number L is such that the values of f are close to the **same number** L for all values of x sufficiently close to a. This implies condition (B) namely, *the values $f(x_1), f(x_2)$ of f are close to* **each other** *for all values x_1, x_2 of x sufficiently close to a.*

Hence the condition (B) *necessarily* holds if the condition (A) holds for some number L. Therefore, if condition (B) is not satisfied, then no number L can exist for which condition (A) is satisfied.

Hence to prove the *non-existence* of the limit of f at a, it is enough to do the following:

Find a *particular* number $\epsilon > 0$, with the following property: Given any $\delta > 0$, two numbers x_1 and x_2 in $\in N'_\delta(a)$ can be found such that

$$|f(x_1) - f(x_2)| \geq \epsilon.$$

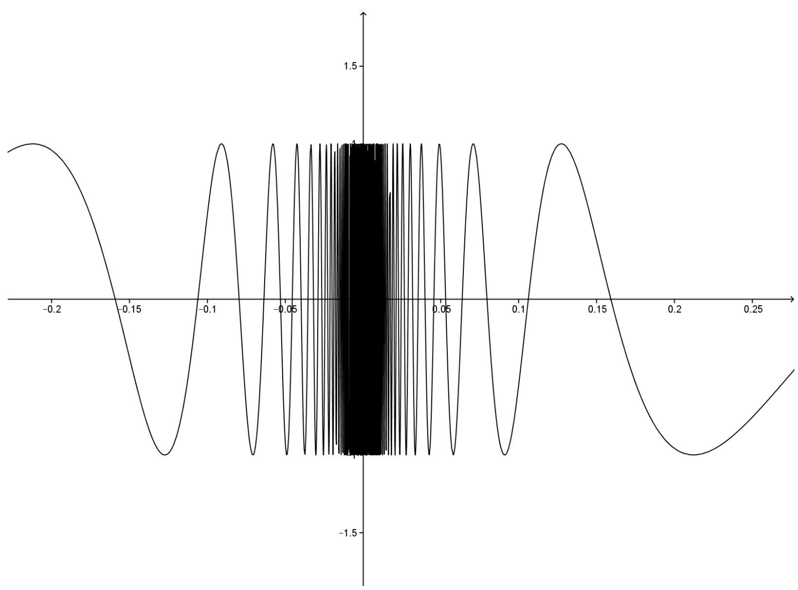
Example : Show that $\lim_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ does not exist.

Solution: To see this, take $\epsilon = 1$, and given any $\delta > 0$, choose a positive integer n such that $1/(2n\pi - (\pi/2)) < \delta$. Let
$x_1 = 1/(2n\pi + (\pi/2))$ and $x_2 = 1/(2n\pi - (\pi/2))$. Then clearly,

$$\begin{aligned} &x_1, x_2 \in (-\delta, \delta) - \{0\} \text{ and} \\ &\qquad |f(x_1) - f(x_2)| = |\sin(2n\pi + (\pi/2)) - \sin(2n\pi - (\pi/2))| \\ &\qquad = |1 - (-1)| = 2 > 1 = \epsilon. \end{aligned}$$

Hence the limit $\lim_{x \to 0} \sin(1/x)$ does not exist.

Actually, we see from the figure below that, as x gets close to 0 from the right or from the left, $f(x)$ does not approach any particular number. But $f(x)$ oscillates between 1 and -1 *infinitely often.* Hence we choose ϵ and x_1, x_2 as in the above.



Right Hand Limit:

Suppose function $f(x)$ is defined for all x such that $a < x < a + \delta_1$ for some $\delta_1 > 0$. We say that right hand limit of f at point $a \in \mathbb{R}$ is $L_1 \in \mathbb{R}$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that
$a < x < a + \delta \Rightarrow |f(x) - L_1| < \epsilon$.

We write this fact as $\lim_{x \to a^+} f(x) = L_1 = \lim_{\substack{x \to a \\ x > a}} f(x)$.

Left Hand Limit:

Suppose function $f(x)$ is defined for all x such that $a - \delta_2 < x < a$ for some $\delta_2 > 0$. We say that left hand limit of f at point $a \in \mathbb{R}$ is $L_2 \in \mathbb{R}$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that
$a - \delta < x < a \Rightarrow |f(x) - L_2| < \epsilon$.

We write this fact as $\lim\limits_{x\to a^-} f(x) = L_2 = \lim\limits_{\substack{x\to a\\ x<a}} f(x)$.

Remark : Observe that $\lim\limits_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x).$$

Example : Let $f(x) = \frac{|x|}{x}$, $x \neq 0$. Find $\lim\limits_{x\to 0} f(x)$, if it exists.

Solution: Note that $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{|x|}{x} = \lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{x}{x} = \lim\limits_{\substack{x\to 0\\ x>0}} 1 = 1$,

and $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} \dfrac{|x|}{x} = \lim\limits_{\substack{x\to 0\\ x<0}} \dfrac{-x}{x} = \lim\limits_{\substack{x\to 0\\ x<0}} (-1) = -1$.

$\therefore \lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$. Hence $\lim\limits_{x\to 0} f(x)$ does not exist.

Properties of Limits:

Theorem : Let f be a function defined in some deleted neighbourhood of a point a. If $\lim\limits_{x\to a} f(x)$ exists then it is unique.

Proof: Let $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} f(x) = L_2$, where $L_1, L_2 \in \mathbb{R}$. We have to prove that $L_1 = L_2$. On the contrary, suppose $L_1 \neq L_2$.

$\therefore L_1 - L_2 \neq 0$. $\quad\quad \therefore |L_1 - L_2| > 0$.

Since $\lim\limits_{x\to a} f(x) = L_1$, we know that
for $\epsilon = |L_1 - L_2| > 0$, $\exists\, \delta_1 > 0$ such that
$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\epsilon}{2}$. (1)

Since $\lim\limits_{x\to a} f(x) = L_2$, we know that
for $\epsilon = |L_1 - L_2| > 0$, $\exists\, \delta_2 > 0$ such that
$0 < |x - a| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\epsilon}{2}$. (2)

Let $\delta = \min\{\delta_1, \delta_2\}$.
$\therefore$ by (1) and (2) we get
$0 < |x - a| < \delta \Rightarrow |f(x) - L_1| < \frac{\epsilon}{2}, |f(x) - L_2| < \frac{\epsilon}{2}$. (3)

Now $\epsilon = |L_1 - L_2| = |L_1 - f(x) + f(x) - L_2|$

$$\begin{aligned} &\le |L_1 - f(x)| + |f(x) - L_2| && \text{[by triangle inequality]} \\ &= |f(x) - L_1| + |f(x) - L_2| && \text{[since } |a-b| = |b-a|\text{]} \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon, \quad \text{if } 0 < |x-a| < \delta, \quad \text{by (3).} \\ &\therefore \epsilon < \epsilon, \end{aligned}$$

which is a contradiction. So $L_1 \neq L_2$ is not possible. Thus $L_1 = L_2$.
$\therefore \lim\limits_{x \to a} f(x)$ is unique, if it exists.

Theorem: Let f and g be two functions such that $\lim\limits_{x \to a} f(x) = K$ and $\lim\limits_{x \to a} g(x) = L$. Then

a) $\lim\limits_{x \to a} \alpha\, f(x) = \alpha \lim\limits_{x \to a} f(x) = \alpha K$ where α is a constant.

b) $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = K \pm L$.

c) $\lim\limits_{x \to a} (f \cdot g)(x) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = K \cdot L$.

d) $\lim\limits_{x \to a} \dfrac{f}{g}(x) = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{K}{L}$, provided $L \neq 0$.

Proof: a) If $\alpha = 0$ then $\alpha f(x) = 0,\ \forall\, x$.
$\therefore \lim\limits_{x \to a} \alpha\, f(x) = \lim\limits_{x \to a} 0 = 0 = 0 \cdot \lim\limits_{x \to a} f(x) = 0\ K$.
Suppose $\alpha \neq 0$.
Let $\epsilon > 0$, we have to find $\delta > 0$, such that
$0 < |x-a| < \delta \Rightarrow |\alpha f(x) - \alpha K| < \epsilon$.

Note that $|\alpha f(x) - \alpha K| = |\alpha|\ |f(x) - K|$. (1)

Since $\lim\limits_{x \to a} f(x) = K$, we know that for $\frac{\epsilon}{|\alpha|} > 0$, $\exists\, \delta_1 > 0$ such that

$0 < |x-a| < \delta_1 \Rightarrow |f(x) - K| < \frac{\epsilon}{|\alpha|}$.

Let $\delta = \delta_1$. Then by (1) we get

$$\begin{aligned} 0 < |x-a| < \delta \Rightarrow |\alpha f(x) - \alpha K| &= |\alpha||f(x) - K| \\ &< |\alpha| \frac{\epsilon}{|\alpha|} = \epsilon. \\ \therefore 0 < |x-a| < \delta &\Rightarrow |\alpha f(x) - \alpha K| < \epsilon. \end{aligned}$$

$\therefore \lim\limits_{x \to a} \alpha\, f(x) = \alpha K = \alpha \lim\limits_{x \to a} f(x)$.

b) Let $\epsilon > 0$, we have to find $\delta > 0$ such that
$0 < |x-a| < \delta \Rightarrow |(f+g)(x) - (K+L)| < \epsilon$.

Consider

$$\begin{aligned} |(f+g)(x)-(K+L)| &= |f(x)+g(x)-K-L| \\ &= |f(x)-K+g(x)-L| \\ &\leq |f(x)-K|+|g(x)-L|, \qquad (1) \end{aligned}$$

by triangle inequality.

Since $\lim\limits_{x\to a} f(x)=K$, we know that for $\frac{\epsilon}{2}>0,\ \exists\,\delta_1>0$ such that
$0<|x-a|<\delta_1 \Rightarrow |f(x)-K|<\frac{\epsilon}{2}.$ (2)

Since $\lim\limits_{x\to a} g(x)=L$, we know that for $\frac{\epsilon}{2}>0,\ \exists\ \delta_2>0$ such that
$0<|x-a|<\delta_2 \Rightarrow |g(x)-L|<\frac{\epsilon}{2}.$ (3)
Let $\delta=\min\{\delta_1,\delta_2\}$.
$\therefore$ by (2) and (3), (1) gives
$0<|x-a|<\delta \Rightarrow |(f+g)(x)-(K+L)|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$
$\therefore \lim\limits_{x\to a}(f+g)(x)=K+L=\lim\limits_{x\to a} f(x)+\lim\limits_{x\to a} g(x).$
Similarly,
$\lim\limits_{x\to a}(f-g)(x)=K-L=\lim\limits_{x\to a} f(x)-\lim\limits_{x\to a} g(x).$
c) First we prove that if $\lim\limits_{x\to a} g(x)=L$ then the function g is bounded in some deleted neighbourhood of the point a.
Since $\lim\limits_{x\to a} g(x)=L$, we know that
for $\epsilon=1>0,\ \exists\ \delta>0$ such that
$0<|x-a|<\delta \Rightarrow |g(x)-L|<\epsilon=1$
i.e. $-1<g(x)-L<1$
i.e. $L-1<g(x)<1+L$
Let $M=\max\{|L-1|,|1+L|\}$.
$\therefore |g(x)|<M,\quad \forall x\in N'_\delta(a).$
$\therefore g(x)$ is bounded in the deleted neighbourhood $N'_\delta(a)$.
Now we prove **c)**.
Let $\epsilon>0$, we have to find $\delta>0$ such that
$0<|x-a|<\delta \Rightarrow |(fg)(x)-KL|<\epsilon.$

Note that, using the triangle inequality, we have

$$\begin{aligned}
|(fg)(x) - KL| &= |f(x)g(x) - g(x)K + g(x)K - KL| \\
&= |g(x)(f(x) - K) + K(g(x) - L)| \\
&\le |g(x)(f(x) - K)| + |K(g(x) - L)| \\
&= |g(x)|\,|f(x) - K| + |K|\,|g(x) - L| \\
&\le |g(x)|\,|f(x) - K| + (|K| + 1)|g(x) - L|. \qquad (1)
\end{aligned}$$

Since $\lim\limits_{x\to a} g(x) = L$, we know that
$g(x)$ is bounded in some deleted neighbourhood of the point a.
$\therefore \exists\ \delta_1 > 0$ and $M \in \mathbb{R}, M > 0$ such that
$|g(x)| < M, \ \ \forall x \in N'_{\delta_1}(a).$ (2)
Since $\lim\limits_{x\to a} f(x) = K$, we know that
for $\frac{\epsilon}{2M} > 0,\ \exists\ \delta_2 > 0$ such that
$0 < |x - a| < \delta_2 \Rightarrow |f(x) - K| < \frac{\epsilon}{2M}.$ (3)
Since $\lim\limits_{x\to a} g(x) = L$, we know that
for $\frac{\epsilon}{2(|K|+1)} > 0,\ \exists\ \delta_3 > 0$ such that
$0 < |x - a| < \delta_3 \Rightarrow |g(x) - L| < \frac{\epsilon}{2(|K|+1)}.$ (4)
Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.
$\therefore$ by using (2), (3) and (4), (1) gives
$0 < |x - a| < \delta \Rightarrow |(fg)(x) - KL| < \frac{\epsilon}{2} + \frac{(|K|+1)\epsilon}{2(|K|+1)} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$
$\therefore 0 < |x - a| < \delta \Rightarrow |(fg)(x) - KL| < \epsilon.$
$\therefore \lim\limits_{x\to a}(fg)(x) = KL = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x).$

A second proof of part c): Let $\epsilon > 0$, we have to find $\delta > 0$ such that
$0 < |x - a| < \delta \Rightarrow |(fg)(x) - KL| < \epsilon.$

Note that

$$
\begin{aligned}
& |(fg)(x) - KL| = |f(x)g(x) - KL| \\
= \quad & |f(x)g(x) - K\,g(x) - L\,f(x) + LK \\
& \qquad + L\,f(x) - LK + K\,g(x) - LK| \\
= \quad & |g(x)(f(x) - K) - L(f(x) - K) + L(f(x) - K) \\
& \qquad\qquad\qquad\qquad + K(g(x) - L)| \\
= \quad & |(f(x) - K)(g(x) - L) + L(f(x) - K) + K(g(x) - L)| \\
\le \quad & |f(x) - K|\,|g(x) - L| + |L|\,|f(x) - K| + |K|\,|g(x) - L| \\
\le \quad & |f(x) - K|\,|g(x) - L| + (|L| + 1)|f(x) - K| \\
& \qquad\qquad\qquad\qquad + (|K| + 1)|g(x) - L|. \qquad (1)
\end{aligned}
$$

Since $\lim\limits_{x\to a} f(x) = K$, we know that
for $\frac{\sqrt{\epsilon}}{3} > 0$, $\exists\ \delta_1 > 0$ such that
$0 < |x - a| < \delta_1 \Rightarrow |f(x) - K| < \frac{\sqrt{\epsilon}}{3}$. (2)

Since $\lim\limits_{x\to a} g(x) = L$, we know that
for $\sqrt{\epsilon} > 0$, $\exists\ \delta_2 > 0$ such that
$0 < |x - a| < \delta_2 \Rightarrow |g(x) - L| < \sqrt{\epsilon}$. (3)

Since $\lim\limits_{x\to a} f(x) = K$, we know that
for $\frac{\epsilon}{3(|L|+1)} > 0$, $\exists\ \delta_3 > 0$ such that
$0 < |x - a| < \delta_3 \Rightarrow |f(x) - K| < \frac{\epsilon}{3(|L|+1)}$. (4)

Since $\lim\limits_{x\to a} g(x) = L$, we know that
for $\frac{\epsilon}{3(|K|+1)} > 0$, $\exists\ \delta_4 > 0$ such that
$0 < |x - a| < \delta_4 \Rightarrow |g(x) - L| < \frac{\epsilon}{3(|K|+1)}$. (5)

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Then using (2), (3), (4) and (5), (1) gives

$$
\begin{aligned}
0 < |x - a| < \delta \quad \Rightarrow \quad & |(fg)(x) - KL| < \frac{\sqrt{\epsilon}}{3}\sqrt{\epsilon} \\
& \qquad + (|L| + 1)\frac{\epsilon}{3(|L| + 1)} + \frac{(|K| + 1)\epsilon}{3(|K| + 1)} \\
= \quad & \epsilon. \\
\therefore 0 < |x - a| < \delta \quad \Rightarrow \quad & |(fg)(x) - KL| < \epsilon. \\
\therefore \lim_{x\to a}(fg)(x) \quad = \quad & KL.
\end{aligned}
$$

d) First we prove that $\lim\limits_{x\to a}\dfrac{1}{g(x)} = \dfrac{1}{L}$.
Let $\epsilon > 0$, we have to find $\delta > 0$ such that
$0 < |x-a| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{L}\right| < \epsilon$.
Note that
$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| = \left|\frac{L-g(x)}{g(x)L}\right| = \frac{|L-g(x)|}{|g(x)|\,|L|} = \frac{|g(x)-L|}{|g(x)|\,|L|}. \qquad (1)$$
Since $\lim\limits_{x\to a} g(x) = L$, we know that
for $\epsilon = \frac{|L|}{2} > 0,\ \exists\ \delta_1 > 0$ such that
$0 < |x-a| < \delta_1 \Rightarrow ||g(x)| - |L|| \le |g(x) - L| < \frac{|L|}{2}$.
$\therefore 0 < |x-a| < \delta_1 \Rightarrow \frac{-|L|}{2} < |g(x)| - |L| < \frac{|L|}{2}$
i.e. $|L| - \frac{|L|}{2} < |g(x)| < |L| + \frac{|L|}{2}$
i.e. $\frac{|L|}{2} < |g(x)| < \frac{3}{2}|L|$.
$\therefore \frac{2}{|L|} > \frac{1}{|g(x)|} > \frac{2}{3|L|}$.
$$\therefore \frac{1}{|g(x)|} < \frac{2}{|L|}. \qquad (2)$$
Since $\lim\limits_{x\to a} g(x) = L$, we know that
for $\frac{|L|^2\epsilon}{2} > 0,\ \exists\ \delta_2 > 0$ such that
$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - L| < \frac{|L|^2}{2}\epsilon. \qquad (3)$$
Let $\delta = \min\{\delta_1, \delta_2\}$.
$\therefore$ using (2) and (3), (1) gives
$0 < |x-a| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{L}\right| < \frac{2}{|L|\,|L|}\frac{|L|^2\epsilon}{2} = \epsilon$.
$\therefore \lim\limits_{x\to a}\dfrac{1}{g(x)} = \dfrac{1}{L}$.
$\therefore$ by part **c)**,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\left(f(x) \times \frac{1}{g(x)}\right)\\ &= \lim_{x\to a} f(x) \cdot \lim_{x\to a}\frac{1}{g(x)} = K\frac{1}{L} = \frac{K}{L}.\end{aligned}$$

Example: If $\lim\limits_{x\to a} f(x) = K$, then $\lim\limits_{x\to a} |f(x)| = |K|$.

Solution: Since $\lim\limits_{x\to a} f(x) = K$, we know that given $\epsilon > 0,\ \exists\ \delta > 0$ such that
$$0 < |x-a| < \delta \Rightarrow |f(x) - K| < \epsilon. \qquad (1)$$
Now we have the inequality $||f(x)| - |K|| \le |f(x) - K|$. Hence by (1),

$0 < |x-a| < \delta \Rightarrow ||f(x)| - |K|| \leq |f(x) - K| < \epsilon$.
So, $\lim_{x\to a} |f(x)| = |K|$.

Example: Evaluate $\lim_{x\to 2}(2x^2 + 3x - 6)$.

Solution: Using the above properties of limits, we get

$$\begin{aligned}\lim_{x\to 2}(2x^2+3x-6) &= 2\lim_{x\to 2} x^2 + 3\lim_{x\to 2} x - \lim_{x\to 2} 6 \\ &= 2\times 2^2 + 3\times 2 - 6 = 8.\end{aligned}$$

Example: Evaluate the following limits:

a) $\lim_{x\to 2}(2x^3 - 3x + 4)$.

Solution: Consider $\lim_{x\to 2}(2x^3 - 3x + 4) = 2\lim_{x\to 2}(x^3) - 3\lim_{x\to 2}(x) + \lim_{x\to 2}(4)$
$= 2(2^3) - 3(2) + (4) = 16 - 6 + 4 = 14$.

b) $\lim_{x\to 1} \dfrac{x^3 + 4x - 3}{x^2 - 12}$.

Solution: Consider $\lim_{x\to 1} \dfrac{x^3 + 4x - 3}{x^2 - 12}$
$= \dfrac{\lim_{x\to 1}(x^3 + 4x - 3)}{\lim_{x\to 1}(x^2 - 12)}$ [since $\lim_{x\to 1}(x^2 - 12) = -11 \neq 0$]
$= \dfrac{1^3 + 4(1) - 3}{1^2 - 12} = \dfrac{2}{-11} = \dfrac{-2}{11}$.

c) $\lim_{x\to 1} \dfrac{x^3 - 1}{x^2 - 1}$.

Solution: Consider $\lim_{x\to 1} \dfrac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$
$= \lim_{x\to 1} \dfrac{(x^2+x+1)}{(x+1)}$
$= \dfrac{(1^2+1+1)}{(1+1)} = \dfrac{3}{2}$.

Note: In above example we cannot substitute $x = 1$ in numerator and denominator at the first step, since we are considering the values of x *near* 1 but *different from* 1. Also, for this reason, we can cancel out the common factor $(x-1)$.

Example: Use $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$ and find $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$, $a \neq 0, b \neq 0$.

Solution: We have

$$\lim_{x\to0}\frac{\sin ax}{\sin bx}=\lim_{x\to0}\left(\frac{\sin ax}{ax}\times\frac{bx}{\sin bx}\times\frac{a}{b}\right)$$
$$=\lim_{x\to0}\frac{\sin ax}{ax}\times\frac{1}{\lim_{x\to0}\frac{\sin bx}{bx}}\times\lim_{x\to0}\frac{a}{b}=1\times\frac{1}{1}\times\frac{a}{b}=\frac{a}{b}.$$

Example: Evaluate $\lim\limits_{x\to4}\dfrac{4-\sqrt{x+12}}{x-4}$.

Solution: We have

$$\begin{aligned}\lim_{x\to4}\frac{4-\sqrt{x+12}}{x-4} &= \lim_{x\to4}\frac{4-\sqrt{x+12}}{x-4}\times\frac{4+\sqrt{x+12}}{4+\sqrt{x+12}}\\ &= \lim_{x\to4}\frac{16-(x+12)}{(x-4)(4+\sqrt{x+12})}\\ &= \lim_{x\to4}\frac{4-x}{(x-4)(4+\sqrt{x+12})}\\ &= \lim_{x\to4}\frac{-1}{4+\sqrt{x+12}}\\ &= \frac{-1}{4+\sqrt{4+12}}=\frac{-1}{8}.\end{aligned}$$

Theorem: The "Sandwich" Theorem. Suppose $g(x)\le f(x)\le h(x)$ for all x in some deleted neighbourhood of a point a.

If $\lim\limits_{x\to a}g(x)=\lim\limits_{x\to a}h(x)=L$ then $\lim\limits_{x\to a}f(x)=L$.

Proof: Let $\epsilon>0$, we have to find $\delta>0$ such that
$0<|x-a|<\delta\Rightarrow|f(x)-L|<\epsilon.$ (1)

Since $\lim\limits_{x\to a}g(x)=L$, we know that
for $\epsilon>0$, $\exists\ \delta_1>0$ such that
$0<|x-a|<\delta_1\Rightarrow|g(x)-L|<\epsilon$
i.e. $L-\epsilon<g(x)<L+\epsilon.$ (2)

Since $\lim\limits_{x\to a}h(x)=L$, we know that
for $\epsilon>0$, $\exists\ \delta_2>0$ such that
$0<|x-a|<\delta_2\Rightarrow|h(x)-2|<\epsilon$
i.e. $L-\epsilon<h(x)<L+\epsilon.$ (3)

Since $g(x)\le f(x)\le h(x)$ for all x in some deleted neighbourhood of a, $\exists\ \delta_3>0$ such that

$0 < |x-a| < \delta_3 \Rightarrow g(x) \leq f(x) \leq h(x)$. (4)
Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.
By (2), (3) and (4) we get
$0 < |x-a| < \delta \Rightarrow L - \epsilon < g(x) \leq f(x) \leq h(x) < L + \epsilon$
i.e. $L - \epsilon < f(x) < L + \epsilon$
i.e. $|f(x) - L| < \epsilon$.
$\therefore \lim\limits_{x \to a} f(x) = L$.

Example: By using sandwich theorem show that $\lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$ for any $a > 0$.

Solution: For any $x > 0$, we have

$$\begin{aligned} |\sqrt{x} - \sqrt{a}| = \left| \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}} \right| &= \frac{|x-a|}{\sqrt{x} + \sqrt{a}} \\ &< \frac{|x-a|}{\sqrt{a}} \qquad [\text{as } \sqrt{x} > 0]. \\ \therefore -\frac{|x-a|}{\sqrt{a}} &\leq \sqrt{x} - \sqrt{a} \leq \frac{|x-a|}{\sqrt{a}}. \end{aligned}$$

Therefore since $\lim\limits_{x \to a} \dfrac{-|x-a|}{\sqrt{a}} = \lim\limits_{x \to a} \dfrac{|x-a|}{\sqrt{a}} = 0$, by sandwich theorem we get $\lim\limits_{x \to a} (\sqrt{x} - \sqrt{a}) = 0$.
$\therefore \lim\limits_{x \to a} \sqrt{x} - \lim\limits_{x \to a} \sqrt{a} = 0$.
$\therefore \lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$.

Example : Evaluate $\lim\limits_{x \to 0} x^2 \cos(1/x)$.

Solution: We know that for all $x \neq 0$, $-1 \leq \cos(1/x) \leq 1$.
$\therefore -x^2 \leq x^2 \cos(1/x) \leq x^2$, as $x^2 > 0$.
Since $\lim\limits_{x \to 0} (-x^2) = \lim\limits_{x \to 0} x^2 = 0$, by sandwich theorem we get
$\lim\limits_{x \to 0} x^2 \cos(1/x) = 0$.

Some Extensions of the Limit Concept:
Infinite Limit: Let f be defined in some deleted neighbourhood of a point a, then $\lim\limits_{x \to a} f(x) = \infty$ if for any $K \in \mathbb{R}$, $\exists\ \delta > 0$ such that
$0 < |x-a| < \delta \Rightarrow f(x) > K$.
Also, $\lim\limits_{x \to a} f(x) = -\infty$ if for any $K \in \mathbb{R}$, $\exists\ \delta > 0$ such that
$0 < |x-a| < \delta \Rightarrow f(x) < K$.

Limit at Infinity: Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$. Suppose $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$.

We say that $\lim\limits_{x\to\infty} f(x) = L$ if for every $\epsilon > 0, \exists\ K \in \mathbb{R}$ such that $x > K \Rightarrow |f(x) - L| < \epsilon$.

Suppose $(-\infty, b) \subseteq A$ for some $b \in \mathbb{R}$. We say that $\lim\limits_{x\to-\infty} f(x) = L$ if for every $\epsilon > 0,\ \exists\ K \in \mathbb{R}$, such that,
$x < K \Rightarrow |f(x) - L| < \epsilon$.

Note: We have introduced the symbols ∞ and $-\infty$ in the above definitions. They are not real numbers.

Note: We will make use of the following results:

1. $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.

Proof: In fact, given any $\epsilon > 0$, choose $K > 1/\epsilon$. Then
$x > K \Rightarrow x > 1/\epsilon \Rightarrow \epsilon > 1/x \Rightarrow 0 < 1/x < \epsilon$.

2. $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$.

Proof: In fact, given any $\epsilon > 0$, choose $K < -1/\epsilon$. Then x is negative and so
$x < K \Rightarrow x < -1/\epsilon \Rightarrow \epsilon > -1/x \Rightarrow -\epsilon < 1/x < 0$.

3. $\lim\limits_{x\to\infty} e^x = \infty$.

Proof: Note that $e = 2.7182... > 1$. Hence e^x increases as x increases. So, let $e = 1 + t$ where $t > 1$. Given any $K > 0$, choose a positive integer n such that $n > K/t$. Then

$$x > n \Rightarrow e^x > (1 + t)^n = 1 + tn + \cdots + t^n > tn > K \Rightarrow e^x > K.$$

Hence $e^x \to \infty$ as $x \to \infty$.

4. $\lim\limits_{x\to-\infty} e^x = 0$.

Proof: Let any $\epsilon > 0$ be given. In the last result, taking $K = 1/\epsilon$, we can choose a positive integer n such that

$$-x > n \Rightarrow e^{-x} > 1/\epsilon \Rightarrow 0 < e^x < \epsilon.$$

Hence $e^x \to 0$ as $x \to -\infty$.

5. $\lim\limits_{x\to 0^+} e^x = \lim\limits_{\substack{x\to 0\\ x>0}} e^x = 1.$ **6.** $\lim\limits_{x\to 0^-} e^x = \lim\limits_{\substack{x\to 0\\ x<0}} e^x = 1$

7. $\lim\limits_{x\to 0^+} e^{\frac{1}{x}} = \lim\limits_{\substack{x\to 0\\ x>0}} e^{\frac{1}{x}} = e^{\infty} = \infty$

8. $\lim\limits_{x\to 0^-} e^{\frac{1}{x}} = \lim\limits_{\substack{x\to 0\\ x<0}} e^{\frac{1}{x}} = \dfrac{1}{e^{\infty}} = \dfrac{1}{\infty} = 0.$

Example: Evaluate $\lim\limits_{x\to\infty} \dfrac{5x^2+3x+20}{3x^2-2x}$.

Solution: We have

$$\lim_{x\to\infty} \frac{5x^2+3x+20}{3x^2-2x} = \lim_{x\to\infty} \frac{(5+\frac{3}{x}+\frac{20}{x^2})}{(3-\frac{2}{x})} = \frac{5+0+0}{3-0} = \frac{5}{3}.$$

Example : If $\lim\limits_{x\to a} f(x) > 0$ then show that there exists a $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow f(x) > 0$.

Solution: Let $\lim\limits_{x\to a} f(x) = L > 0$.
$\therefore$ for $\epsilon = \frac{L}{2} > 0,\ \exists\ \delta > 0$ such that
$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$
i.e. $|f(x) - L| < \frac{L}{2}$
i.e. $-\frac{L}{2} < f(x) - L < \frac{L}{2}$
i.e. $L - \frac{L}{2} < f(x) < L + \frac{L}{2}$
i.e. $\frac{L}{2} < f(x) < \frac{3L}{2}$.
$\therefore f(x) > \frac{L}{2} > 0$.
$\therefore 0 < |x-a| < \delta \Rightarrow f(x) > 0$.

Exercises:

1. Evaluate following limits, if they exist.
   (a) $\lim\limits_{x\to 1+} \dfrac{1}{x-1}$ (b) $\lim\limits_{x\to\infty} \dfrac{x+2}{\sqrt{x}}$ $(x > 0)$
   (c) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x}-5}{\sqrt{x}+3}$ $(x > 0)$ (d) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x}-x}{\sqrt{x}+x}$ $(x > 0)$
2. If $\lim\limits_{x\to a} f(x) < 0$ then show that there exists a $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow f(x) < 0$.

Continuity

The idea of continuity may be thought of informally as the quality of having parts that are in immediate connection with one another. The idea evolved from the intuitive notion of a curve without breaks or jumps.

Defintion: Let f be a function defined in some neighbourhood of a point a. We say that the function f is *continuous* at the point a if

$$\lim_{x \to a} f(x) = f(a).$$

That is, f is continuous at a if for every $\epsilon > 0$, $\exists\, \delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

That is, f is continuous at a if following conditions holds:
(i) $f(a)$ exists, i.e. f is defined at the point a and
(ii) $\lim\limits_{x \to a} f(x)$ exists and
(iii) $\lim\limits_{x \to a} f(x) = f(a)$.

If any one of the above conditions is not satisfied then f is not continuous (or is *discontinuous*) at the point a.

If conditions (i) and (ii) are satisfied but (iii) is not satisfied then we say that f has a *removable* discontinuity at the point a. This discontinuity can be removed by defining $f(a)$ to be the value of the limit of f at a.

If condition (ii) is not satisfied then we say that f has a *non-removable* discontinuity or an *essential* discontinuity at the point a.

Example: Let $f(x) = (x^2 - 1)/(x - 1), \quad x \neq 1$. This function is not defined at 1, and is therefore discontinuous at 1. But the limit of f at 1 exists and in fact, $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1}(x + 1) = 2$. So, if we *define* $f(1)$ to be 2, then f becomes continuous at 1. So f has a removable discontinuity at 1.

Example: $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} f(x) &= \tfrac{1}{x}, \quad x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

Here $f(0) = 0$, but $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist. Hence f has a non-removable discontinuity at $x = 0$.

If $a \in \mathbb{R}$, $a \neq 0$ then $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1}{x} = \frac{1}{a} = f(a)$. So f is continuous at $x = a$. Thus f is continuous everywhere except at 0.

Example: Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} f(x) &= \tfrac{|x|}{x}, \quad x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

Here $f(0) = 0$. Now $\lim_{x \to 0^+} f(x) = \lim_{\substack{x \to 0 \\ x > 0}} \frac{|x|}{x} = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$,

and $\lim_{x \to 0^-} f(x) = \lim_{\substack{x \to 0 \\ x < 0}} \frac{|x|}{x} = \lim_{\substack{x \to 0 \\ x < 0}} \frac{-x}{x} = \lim_{x \to 0} (-1) = -1$.

$\therefore \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$.

$\therefore \lim_{x \to 0} f(x)$ does not exist.

$\therefore f$ is discontinuous at $x = 0$.

'0' is a point of essential discontinuity of f.

Note that f can also be defined as

$$f(x) = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -1 & , \quad x < 0 \end{cases}$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} f(x) &= \tfrac{\sin x}{x}, \quad x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

Here $f(0) = 0$.

Also, we know that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. So $\lim_{x \to 0} f(x) = 1 \neq 0 = f(0)$.

$\therefore f$ is discontinuous at $x = 0$.

It is a removable discontinuity. We can remove this discontinuity by redefining $f(0)$ to be 1.

Example: Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} f(x) &= \sin\left(\tfrac{1}{x}\right), \quad x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

We know $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

$\therefore f$ has an essential discontinuity at $x = 0$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} f(x) &= x\sin\tfrac{1}{x}, \quad x \neq 0 \\ &= \quad 0, \qquad x = 0. \end{aligned}$$

Note that $0 \leq |x\sin\frac{1}{x}| = |x|\cdot|\sin\frac{1}{x}| \leq |x|$ because $|\sin\frac{1}{x}| \leq 1$. Therefore, since $\lim\limits_{x\to 0}(0) = \lim\limits_{x\to 0}|x| = 0$, by sandwich theorem it follows that $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$. Since $f(0) = 0$, we see that f is continuous at $x = 0$.

Example : Discuss continuity of f at $x = 0$, where

$$\begin{aligned} f(x) &= \tfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, \quad x \neq 0 \\ &= \quad 0, \qquad x = 0. \end{aligned}$$

Solution: Here $f(0) = 0$. Note that

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{\substack{x\to 0\\ x>0}} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \\ &= \lim_{\substack{x\to 0\\ x>0}} \frac{e^{\frac{1}{x}}\left(1-\frac{1}{e^{\frac{1}{x}}}\right)}{e^{\frac{1}{x}}\left(1+\frac{1}{e^{\frac{1}{x}}}\right)} = \lim_{\substack{x\to 0\\ x>0}} \frac{1-\frac{1}{e^{\frac{1}{x}}}}{1+\frac{1}{e^{\frac{1}{x}}}} = \frac{1-0}{1+0} = 1. \end{aligned}$$

$$\text{Next,} \quad \lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0\\ x<0}} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \frac{0-1}{0+1} = -1.$$

Thus $\lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$. Hence $\lim\limits_{x\to 0} f(x)$ does not exist. So, f is discontinuous at $x = 0$.

Example : $f(x) = \frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, x \neq 0, f(0) = 0.$

Discuss continuity of f at $x = 0$.

Solution: Here $f(0) = 0$. Note that

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{\substack{x\to 0\\ x>0}} \frac{x\,e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} \\ &= \lim_{\substack{x\to 0\\ x>0}} \frac{e^{\frac{1}{x}}x}{e^{\frac{1}{x}}\left(1+\frac{1}{e^{\frac{1}{x}}}\right)} = \lim_{\substack{x\to 0\\ x>0}} \frac{x}{\frac{1}{e^{\frac{1}{x}}}+1} = \frac{0}{0+1} = 0, \\ \text{and} \quad \lim_{x\to 0^-} f(x) &= \lim_{\substack{x\to 0\\ x<0}} \frac{x\,e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = \frac{0\times 0}{1+0} = 0. \end{aligned}$$

$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$.
$\therefore$ the function f is continuous at $x = 0$.

Example : Discuss continuity of f at the points 1, 2 and 4 where

$$f(x) = \begin{cases} 2x-1 & , \quad x \le 1 \\ x^2 & , \quad 1 < x < 2 \\ 3x-4 & , \quad 2 \le x < 4 \\ x^{3/2} & , \quad x \ge 4 \end{cases}$$

Solution: (i) Here $f(1) = 2(1) - 1 = 2 - 1 = 1$. Now

$$\lim_{x\to 1^+} f(x) = \lim_{\substack{x\to 1 \\ x>1}} f(x) = \lim_{\substack{x\to 1 \\ x>1}} x^2 = (1)^2 = 1$$

and

$$\lim_{x\to 1^-} f(x) = \lim_{\substack{x\to 1 \\ x<1}} f(x) = \lim_{\substack{x\to 1 \\ x<1}} (2x-1) = 2(1) - 1 = 2 - 1 = 1.$$

$\therefore \lim_{x\to 1} f(x) = 1 = f(1)$.
$\therefore$ the function f is continuous at $x = 1$.
(ii) Here $f(2) = 3(2) - 4 = 6 - 4 = 2$.
Also, $\lim_{x\to 2^+} f(x) = \lim_{\substack{x\to 2 \\ x>2}} f(x) = \lim_{\substack{x\to 2 \\ x>2}} (3x-4) = 3(2) - 4 = 6 - 4 = 2$

and $\lim_{x\to 2^-} f(x) = \lim_{\substack{x\to 2 \\ x<2}} f(x) = \lim_{\substack{x\to 2 \\ x<2}} x^2 = 2^2 = 4$.

$\therefore \lim_{x\to 2} f(x)$ does not exist.
$\therefore f$ is not continuous at $x = 2$.
(iii) Here $f(4) = 4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$.

Note that $\lim_{x\to 4^+} f(x) = \lim_{\substack{x\to 4 \\ x>4}} f(x) = \lim_{\substack{x\to 4 \\ x>4}} x^{\frac{3}{2}} = 4^{\frac{3}{2}} = 8$

and $\lim_{x\to 4^-} f(x) = \lim_{\substack{x\to 4 \\ x<4}} f(x) = \lim_{\substack{x\to 4 \\ x<4}} (3x-4) = 3(4) - 4 = 12 - 4 = 8$.

$\therefore \lim_{x\to 4} f(x) = 8 = f(4)$.
$\therefore$ the function f is continuous at $x = 4$.

Example: Find numbers α and β if the function f is continuous at every point of $(-3, 5)$, where

$$f(x) = \begin{cases} x+\alpha & , \quad -3 < x < 1 \\ 3x+2 & , \quad 1 \le x < 3 \\ \beta + x & , \quad 3 \le x < 5 \end{cases}$$

Solution: Here $f(1) = 3(1) + 2 = 3 + 2 = 5$.

Now $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{\substack{x \to 1 \\ x<1}} f(x) = \lim\limits_{\substack{x \to 1 \\ x<1}} (x+\alpha) = 1 + \alpha$.

Since f is continuous at $x = 1$, we have
$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1} f(x) = f(1)$.
$\therefore 1 + \alpha = 5. \qquad \therefore \alpha = 5 - 1 = 4$.

Also, $f(3) = \beta + 3$, and
$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{\substack{x \to 3 \\ x<3}} f(x) = \lim\limits_{\substack{x \to 3 \\ x<3}} (3x+2) = 3(3) + 2 = 9 + 2 = 11$.

Since f is continuous at $x = 3$, we have
$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3} f(x) = f(3)$.
$\therefore \beta + 3 = 11. \qquad \therefore \beta = 11 - 3 = 8$.
$\therefore \alpha = 4, \beta = 8$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$, and $K > 0$ be such that
$|f(x) - f(y)| \le K|x - y| \qquad \forall\, x, y \in \mathbb{R}$.
Show that f is continuous at every point $c \in \mathbb{R}$.

Solution: Let $c \in \mathbb{R}$. By data,
$|f(x) - f(y)| \le K|x - y|, \ \ \forall\, x, y \in \mathbb{R}. \qquad (1)$
Let $\epsilon > 0$, we have to find $\delta > 0$ such that
$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

Let $\delta = \frac{\epsilon}{K}$. Then by (1),
$|x - c| < \delta \Rightarrow |f(x) - f(c)| \le K|x - c| < K\delta = \epsilon$.
$\therefore \lim\limits_{x \to c} f(x) = f(x)$.
$\therefore f$ is continuous at $c \in \mathbb{R}$. So, f is continuous on $\mathbb{R}$.

Exercises:

1. Discuss continuity of the function f at $x = 0$:

   (i) $f(x) = \frac{e^{\frac{1}{x^2}}}{1+x} \qquad x \ne -1$
   (ii) $f(x) = \frac{1}{x} \sin \frac{1}{x}, x \ne 0, f(0) = 0$
   (iii) $f(x) = \frac{x-|x|}{x}, x \ne 0, f(0) = 0$.

2. Discuss continuity of f at $x = \frac{1}{2}$, where

$$f(x) = \begin{cases} x & , \quad 0 \le x < \frac{1}{2} \\ 1 & , \quad x = \frac{1}{2} \\ 1-x & , \quad \frac{1}{2} < x < 1. \end{cases}$$

3. Find numbers α and β if the function f is continuous at every point in $(-2, 2)$, where
$$f(x) = \begin{cases} x+\alpha & , \quad -2 < x < 0 \\ 2x+1 & , \quad 0 \le x < 1 \\ \beta - x & , \quad 1 \le x < 2. \end{cases}$$

4. Discuss continuity of the function f at $x = 4$, where
$$f(x) = \begin{cases} \frac{x^2}{4} - 4 & , \quad 0 < x < 4 \\ 0 & , \quad x = 4 \\ 4 - \frac{64}{x^2} & , \quad x > 4. \end{cases}$$

Theorem: If f and g are continuous functions at point c then
(i) αf is continuous at c, for any number α.
(ii) $f \pm g$ is continuous at c.
(iii) $f \cdot g$ is continuous at c.
(iv) f/g is continuous at c, if $g(c) \neq 0$.

Proof: (ii) Since f and g are continuous at point c, we have $\lim_{x\to c} f(x) = f(c), \lim_{x\to c} g(x) = g(c)$. Hence

$$\lim_{x\to c}(f+g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c) = (f+g)(c).$$

$\therefore f + g$ is continuous at point c. Similar proofs for (i), (iii) and (iv) hold.

Theorem: If f is continuous at $x = c$ and g is continuous at $f(c)$ then the composite function gof is continuous at c.

Proof: Since g is continuous at $f(c)$, given $\epsilon > 0$, $\exists\, \delta_1 > 0$ such that
$|f(x) - f(c)| < \delta_1 \Rightarrow |(gof)(x) - (gof)(c)| < \epsilon$ (1)

Since f is continuous at c,
for $\delta_1 > 0$, $\exists\ \delta > 0$ such that
$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1.$ (2)

By (1) and (2) we get
$|x - c| < \delta \Rightarrow |(gof)(x) - (gof)(c)| < \epsilon.$
$\therefore gof$ is continuous at c.

Example: If a function f is continuous at point c then function $|f|$ is also continuous at point c.

Solution: Let $\epsilon > 0$, we have to find $\delta > 0$ such that
$|x-c| < \delta \Rightarrow ||f|(x) - |f|(c)| < \epsilon$.
Since f is continuous at c, we know that
given $\epsilon > 0, \quad \exists \quad \delta_1 > 0$ such that
$|x-c| < \delta_1 \Rightarrow |f(x) - f(c)| < \epsilon$.
Let $\delta = \delta_1$.
Then, since $||f(x)| - |f(c)|| \leq |f(x) - f(c)|$, it follows that
$|x-c| < \delta \Rightarrow ||f(x)| - |f(x)|| < \epsilon$.
$\therefore |f|$ is continuous at point c.

Example: Determine the points of continuity of the function $h : \mathbb{R} - \{0\} \to \mathbb{R}$, defined by $h(x) = \frac{1+|\sin x|}{x}$.
Solution: Let $f_1(x) = 1$, for all $x \in \mathbb{R}$, $f_2(x) = \sin x$, for all $x \in \mathbb{R}, f_3(x) = |x|$, for all $x \in \mathbb{R}$, $f_4(x) = x$, for all $x \in \mathbb{R}$. We know all these functions are continuous on $\mathbb{R}$. We can write function h as $h = \frac{f_1 + (f_3 o f_2)}{f_4}$. Since addition of continuous functions, division of continuous functions (non-zero denominator) and composition of continuous functions yield a continuous function, we see that h is continuous function on $\mathbb{R} - \{0\}$.

Example: If f is continuous at a point c and $f(x) \geq 0$ in some neighbourhood of c then $\sqrt{f(x)}$ is also continuous at c.

Solution: Let $\epsilon > 0$, we have to find $\delta > 0$ such that
$|x-c| < \delta \Rightarrow |\sqrt{f(x)} - \sqrt{f(c)}| < \epsilon$.
Suppose $f(c) > 0$. Then

$$\begin{aligned} |\sqrt{f(x)} - \sqrt{f(c)}| &= \left| (\sqrt{f(x)} - \sqrt{f(c)}) \times \frac{(\sqrt{f(x)} + \sqrt{f(c)})}{\sqrt{f(x)} + \sqrt{f(c)}} \right| \\ &= \frac{|f(x) - f(c)|}{\sqrt{f(x)} + \sqrt{f(c)}} \leq \frac{|f(x) - f(c)|}{\sqrt{f(c)}}. \qquad (1) \end{aligned}$$

Since $\lim_{x \to c} f(x) = f(c) \quad$ (as f is continuous at $x = c$), we know that
for $\sqrt{f(c)}\epsilon > 0, \quad \exists \quad \delta_1 > 0$ such that
$|x-c| < \delta_1 \Rightarrow |f(x) - f(c)| < \epsilon \cdot \sqrt{f(c)}. \qquad (2)$
Let $\delta = \delta_1$.
$\therefore$ by (1) and (2) we get

$|x-c|<\delta \Rightarrow |\sqrt{f(x)}-\sqrt{f(c)}| < \frac{\epsilon\sqrt{f(c)}}{\sqrt{f(c)}} = \epsilon$
i.e. $|\sqrt{f(x)}-\sqrt{f(c)}| < \epsilon$.
$\therefore \sqrt{f(x)}$ is continuous at c.
Suppose $f(c)=0$.
Let $\epsilon>0$, we have to find $\delta>0$ such that
$|x-c|<\delta \Rightarrow |\sqrt{f(x)}-0| < \epsilon$
i.e. $\sqrt{f(x)} < \epsilon$.
Since f is continuous at c, we know that
for $\epsilon^2>0, \quad \exists \quad \delta_1>0$ such that
$|x-c|<\delta_1 \Rightarrow |f(x)-0|<\epsilon^2$.
i.e. $f(x)<\epsilon^2$
$\therefore \sqrt{f(x)}<\epsilon$.
Let $\delta=\delta_1$.
$\therefore |x-c|<\delta \Rightarrow \sqrt{f(x)}<\epsilon$.
$\therefore \sqrt{f(x)}$ is continuous at c.

Example: Let $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^n, n\in\mathbb{N}$, then f is continuous at any $a\in\mathbb{R}$.

Example: Polynomial function $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, $a_i\in\mathbb{R}$, n non-negative integer, is continuous at any $a\in\mathbb{R}$.

Example: Rational function $r(x)=\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions, is continuous at all points where it is defined i.e. at all points x such that $q(x)\neq 0$.

Example: $f(x)=\frac{\sin x}{\cos x}=\tan x$ is continuous everywhere in $\mathbb{R}$ except on the set $\{\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\cdots\}$ i.e. on the set of zeros of $\cos x$.

Example: Discuss continuity of $\sqrt{(x-2)(x-4)}$.

Solution: $(x-2)(x-4)$ is a polynomial function and hence is continuous everywhere on $\mathbb{R}$.
$\therefore \sqrt{(x-2)(x-4)}$ is continuous $\forall\, x\in\mathbb{R}$ such that $(x-2)(x-4)\geq 0$
i.e. $x-2\geq 0$ and $x-4\geq 0$ or $x-2\leq 0$ and $x-4\leq 0$
i.e. $x\geq 2$ and $x\geq 4$ or $x\leq 2$ and $x\leq 4$
i.e. $x\geq 4$ or $x\leq 2$.
$\therefore \sqrt{(x-2)(x-4)}$ is continuous $\forall\, x\in(-\infty,2]\cup[4,\infty)$.

Example: Let $u(x)=x$ and

$$f(x) = \begin{cases} 0 & , \quad x \neq 0 \\ 1 & , \quad x = 0. \end{cases}$$

Show that $\lim\limits_{x\to 0} f(u(x)) \neq f(\lim\limits_{x\to 0} u(x))$.

Solution: Observe that $f(u(x)) = f(x)$.

$\therefore \lim\limits_{x\to 0} f(u(x)) = \lim\limits_{x\to 0} f(x) = 0$ [since $\lim\limits_{x\to 0^+} f(x) = 0 = \lim\limits_{x\to 0^-} f(x)$.]

Now $\lim\limits_{x\to 0} u(x) = \lim\limits_{x\to 0} x = 0$.

$\therefore f\left(\lim\limits_{x\to 0} u(x)\right) = f(0) = 1$.

$\therefore \lim\limits_{x\to 0} f(u(x)) \neq f\left(\lim\limits_{x\to 0} u(x)\right)$.

Example: Let $f(x) = \frac{x}{|x|}, x \neq 0, f(0) = 0$ then f is continuous everywhere in $\mathbb{R}$ except $x = 0$.

Example: Let

$$f(x) = \begin{cases} 1 & , \quad x \in [-1, 1] \\ 0 & , \quad x \in (-\infty, -1) \cup (1, \infty) \end{cases}$$

then f is continuous everywhere in $\mathbb{R}$ except two points $x = -1$ and $x = 1$.

Example: Let

$$f(x) = \begin{cases} 1 & , \quad x \in (-1, 1) \cup (2, 3) = A \\ 4 - x & , \quad x \in (3, 4) = B \\ 0 & , \quad x \in (A \cup B)^c = \mathbb{R} - (A \cup B), \end{cases}$$

then f is continuous everywhere in $\mathbb{R}$ except three points $x = -1, 1, 2$.

Example: Let

$$f(x) = \begin{cases} n - 1 & , \quad n - 1 < x < n \\ n & , \quad x = n \\ n & , \quad n < x < n + 1, \quad n \in \mathbb{Z} \end{cases}$$

then f is continuous at every point in $\mathbb{Z}^c$. That is, f is continuous everywhere except integer points.

Exercises:

1. Discuss the continuity of the following functions:
   i) $f(x) = \frac{x^2+2x+1}{x^2+1}$ $\quad (x \in \mathbb{R})$
   ii) $f(x) = \sqrt{x + \sqrt{x}}$ $\quad (x \in \mathbb{R}, x \geq 0)$
   iii) $f(x) = \cos(\sqrt{1 + x^2})$, $\quad (x \in \mathbb{R})$
   iv) $f(x) = x - [x]$, $\quad (x \in \mathbb{R})$
   where $[x]$ denotes the greatest integer function.

2. Find two functions f and g such that f is discontinuous at $x = 1$ but $f \cdot g$ is continuous at $x = 1$.

3. Give examples of functions f and g both of which are discontinuous at point $a \in \mathbb{R}$ but for which $f + g$ is continuous at a.

4. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but for which $|f|$ is continuous on $[0, 1]$.

Example: Discuss continuity of the function f on $\mathbb{R}$ where,
$f(x) = 1$ if x is rational and,
$= -1$ if x is irrational.
Solution: Let $c \in \mathbb{R}$. We show that function f is discontinuous at the point c, by showing that $\lim\limits_{x\to c} f(x)$ does not exist.

Let $\epsilon = 1 > 0$ and $\delta > 0$ be any real number. By density theorem, there is a rational number x_1 and an irrational number x_2 in $N'_\delta(c)$. Hence, $|f(x_1) - f(x_2)| = |1 - (-1)| = 2 > 1$.

Hence $\lim\limits_{x\to c} f(x)$ does not exist. Therefore function f is not continuous at point c. Since c be arbitrary real number, f is discontinuous on $\mathbb{R}$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for every rational number x. Show that $f(x) = 0$ for any real number x.
Solution: Let c be any real number. If c is rational number we are through. Suppose c is an irrational number. We will prove that $f(c) = 0$. On the contrary suppose $f(c) \neq 0$. Therefore $|f(c)| > 0$. Let $\epsilon = \frac{|f(c)|}{2}$. Since f is continuous at point c, there is $\delta > 0$ such that $|f(x) - f(c)| < \frac{|f(c)|}{2}$ for all $x \in N_\delta(c)$.
By density theorem there is a rational number $x_1 \in N_\delta(c)$. Hence, $|f(x_1) - f(c)| < \frac{|f(c)|}{2}$, that is $|0 - f(c)| < \frac{|f(c)|}{2}$.
By canceling $|f(c)|$ from both sides we get, $1 < \frac{1}{2}$, which is contradiction.
Hence our assumption $f(c) \neq 0$ is wrong. Therefore $f(c) = 0$.

Example: Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) = g(x)$ for every rational number x. Show that $f(x) = g(x)$ for any real number x.
Solution: Since $f(x) = g(x)$ for every rational number x, therefore $f(x) - g(x) = 0$ for every rational number x.
Therefore $(f - g)(x) = 0$ for every rational number x. Since f, g are continuous functions on $\mathbb{R}$, therefore $f - g$ is continuous on $\mathbb{R}$. Hence by previous example

$(f-g)(x)=0$ for any real number x. Therefore $f(x)=g(x)$ for any real number x.

Note: 1) Let $f,g:\mathbb{R}\to\mathbb{R}$ be continuous functions such that $f(x)=g(x)$ for every irrational number x. Show that $f(x)=g(x)$ for any real number x.

2)Let $f,g:\mathbb{R}\to\mathbb{R}$ are functions such that $f(x)=g(x)$ for every rational number x. If f and g are continuous at a point c then $f(c)=g(c)$.

3)Let $f,g:\mathbb{R}\to\mathbb{R}$ are functions such that $f(x)=g(x)$ for every irrational number x. If f and g are continuous at a point c then $f(c)=g(c)$.

Example: Let $g:\mathbb{R}\to\mathbb{R}$ be a function defined as,
$g(x)=2x$, if x is rational and
$g(x)=x+3$, if x is irrational.
Find all points at which g is continuous.

Solution: Let g be continuous at a point c.
Case1) c is a rational number.
Let $h(x)=x+3$ for all $x\in\mathbb{R}$. Clearly h is continuous function at point c. Also $h(x)=g(x)$ for all irrational numbers x, therefore by note 3) $h(c)=g(c)$. That is $c+3=2c$, hence $c=3$.
Case 2) c is an irrational number.
Let $f(x)=2x$ for all $x\in\mathbb{R}$. Clearly f is continuous function at point c. Also $f(x)=g(x)$ for all rational numbers x, therefore by note 2) $f(c)=g(c)$. That is $2c=c+3$, hence $c=3$, which is a contradiction since c is an irrational number.
Therefore function g is continuous only at $c=3$.

Example: Find all possible continuous functions f on $\mathbb{R}$ such that, $f(x+y)=f(x)f(y),\ \ \forall x,y\ \in\mathbb{R}$.
Solution: Since $f(0)=f(0+0)=f(0)f(0)=(f(0))^2$, therefore $f(0)=0\ or\ f(0)=1$.
Case 1) $f(0)=1$.
Since $f(0)=f(1+(-1))=f(1)f(-1)$, therefore $f(-1)=\frac{1}{f(1)}=(f(1))^{-1}$.
Let $\frac{p}{q}$ be a rational number where p,q are integers, $q>0$.
Consider $f(\frac{p}{q})=f(\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+...+\frac{1}{q}$ ($|p|$ times))
$=f(\frac{1}{q})\ f(\frac{1}{q})\ f(\frac{1}{q})\ ...\ f(\frac{1}{q})$ ($|p|$ times)
$=[f(\frac{1}{q})]^p=[(f(\frac{1}{q}))^q]^{\frac{p}{q}}=[f(\frac{1}{q}\cdot q)]^{\frac{p}{q}}=(f(1))^{\frac{p}{q}}$.
Therefore $f(x)=(f(1))^x$ for every rational number x.

Hence by previous example $f(x) = (f(1))^x$ for any real number x.
Case 2) $f(0) = 0$.
For any real number x, $f(x) = f(x+0) = f(x)f(0) = 0$.

Example: Discuss the continuity of the function
$h(x) = \max\{f(x), g(x)\}$, $\forall\, x \in \mathbb{R}$, where f and g are continuous functions on $\mathbb{R}$.
Solution: We know for any two real numbers a, b, $\max\{a, b\} = \frac{a+b+|a-b|}{2}$.
Therefore for any real number x,
$h(x) = \max\{f(x), g(x)\} = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$
$= \frac{(f+g)(x)+|(f-g)(x)|}{2}$.
Since f, g are continuous on $\mathbb{R}$, $f+g, |f-g|$ are continuous on $\mathbb{R}$.
Hence h is continuous on $\mathbb{R}$.

Example: Discuss continuity of the function f at $x = 0$ where,
$f(x) = \sin(\frac{1}{x})$, $x \neq 0$, and $f(0) = 0$.

Solution: Let $\epsilon = 0.5 > 0$ and $\delta > 0$ be any real number. By Archimedian property choose $n, m \in \mathbb{N}$ such that, $x_1 = \frac{1}{(2n+1)\frac{\pi}{2}} < \delta$ and $x_2 = \frac{1}{m\pi} < \delta$.
Then, $|f(x_1) - f(x_2)| = |(-1)^n - 0| = 1 > 0.5$.
So $\lim\limits_{x \to 0} f(x)$ does not exist. Therefore f is discontinuous at the point 0.

Exercises:

1. Discuss the continuity of following functions.
1) For $x \in \mathbb{R}$, $f(x) = x$ if x is rational; $f(x) = 1 - x$ if x is irrational.
2) $h = \min\{f, g\}$, where f and g are continuous functions on $\mathbb{R}$.

2. Find all possible continuous functions f on $\mathbb{R}$ in each case:
1)$f(xy) = f(x)f(y), \quad \forall x, y \in \mathbb{R}$.
2)$f(x+y) = f(x) + f(y), \quad \forall x, y \in \mathbb{R}$.
3)$f(xy) = f(x) + f(y), \quad \forall x, y \in \mathbb{R}$.

Continuous Functions on Intervals

Let $A \subseteq \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. We say that f is continuous on set A if f is continuous at every point of A.

Definition: A function $f : A \to \mathbb{R} \quad (A \subseteq \mathbb{R})$ is said to be bounded on the set A if these exists $M > 0$ such that $|f(x)| \leq M \quad \forall\, x \in A$.

Note: Let $A \subseteq \mathbb{R}$, $\quad f : A \to \mathbb{R}$, then the Range of f or the image of A under

f is the set $\{f(x) \in \mathbb{R} \mid x \in A\}$. Thus a function f is bounded if and only if its range is a bounded set.

Example: Let $f : (0, 1) \to \mathbb{R},\ f(x) = \frac{1}{x}$, then f is not bounded on $(0, 1)$.

To see this, let $M > 1$ be any given number. Let $x = 1/2M$. Then clearly, $x \in (0, 1)$ and $1/x = 2M > M$ so that $|f(x)| = 1/x > M$.

Example: Let $f : (1, 2) \to \mathbb{R},\ f(x) = \frac{1}{x}$, then f is bounded on $(1, 2)$, since $|f(x)| \leq 1,\ \ \forall\, x \in (1, 2)$.

Example: Let $f : (1, \infty) \to \mathbb{R},\ f(x) = \frac{1}{x}$, then f is bounded on $(1, \infty)$ since $|f(x)| \leq 1,\ \forall\, x \in (1, \infty)$.

Example: Let $f : (0, 1] \to \mathbb{R},\ f(x) = e^{\frac{1}{x}}$, then f is not bounded on $(0, 1]$.

Example: Let $f : [1, \infty) \to \mathbb{R},\ f(x) = e^{\frac{1}{x}}$, then f is bounded on $[1, \infty)$ since $|f(x)| \leq e,\ \forall x \in [1, \infty)$.

Definition : Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. The function f is said to have an absolute maximum value on the set S if there is at least one point c in S such that $f(x) \leq f(c),\ \ \forall x \in S$. The number $f(c)$ is called the absolute maximum value of f on S.

We say that f has an absolute minimum value on S if there is a point $d \in S$ such that $f(x) \geq f(d),\ \forall x \in S$, and then $f(d)$ is called the absolute minimum value of f on S.

Theorem: Boundedness Theorem For Continuous Functions.
Let a function f be continuous on a closed and bounded interval $[a, b]$. Then f is bounded on $[a, b]$.

Proof: We prove the result by contradiction. Assume that f is unbounded (i.e. not bounded) on $[a, b]$.

Since $[a, b] = [a, c] \cup [c, b]$ where $\frac{b+a}{2} = c$ is the midpoint of $[a, b]$, and since f is unbounded on $[a, b]$, we see that f is unbounded on $[a, c]$ or $[c, b]$. Denote by $[a_1, b_1]$ the subinterval on which f is unbounded; if f is unbounded on both the subintervals, let $[c, b] = [a_1, b_1]$.

Since $[a_1, b_1] = [a_1, c_1] \cup [c_1, b_1]$ where $\frac{b_1+a_1}{2} = c_1$ is the midpoint of $[a_1, b_1]$, and since f is unbounded on $[a_1, b_1]$, we see that f is unbounded on $[a_1, c_1]$ or $[c_1, b_1]$. Denote by $[a_2, b_2]$ the subinterval on which f is unbounded; if f is unbounded on both the subintervals, let $[c_1, b_1] = [a_2, b_2]$.

Continuing in this way, let, for each n, $[a_{n+1}, b_{n+1}]$ denote that half of

$[a_n, b_n]$ on which f is unbounded. Note that the length of $[a_n, b_n] = \frac{b-a}{2^n}$, and $a \le a_1 \le a_2 \le \cdots \le a_n \le b_n \le \cdots \le b_2 \le b_1 \le b$, for all n.

Let $A = \{a, a_1, a_2, \cdots\}$. Then $A \subseteq [a, b]$, so that A is a non-empty bounded set of real numbers. Hence the supremum of A exists. Let $\alpha = \sup A$. Clearly, $a \le \alpha \le b$.

(i) Let $a < \alpha < b$.

Since f is continuous at α, for $\epsilon = 1$, $\exists\, \delta > 0$ such that
$|x - \alpha| < \delta \Rightarrow |f(x) - f(\alpha)| < 1$.

Now $|f(x)| = |f(x) - f(\alpha) + f(\alpha)| \le |f(x) - f(\alpha)| + |f(\alpha)|$.

Hence $|x - \alpha| < \delta \Rightarrow |f(x)| < 1 + |f(\alpha)|$.
So, f is bounded on $I = (\alpha - \delta, \alpha + \delta)$.

Here $a \le a_1 \le a_2 \le \cdots \le a_n \le \cdots$. Since $\alpha = \sup A$, corresponding to the above δ, there exists m such that $a_m > \alpha - \delta$. Now choose $n \ge m$ such that $b_n - a_n = \frac{b-a}{2^n} < \delta$. Then $\alpha - \delta < a_m \le a_n < b_n < a_n + \delta \le \alpha + \delta$. Hence $[a_n, b_n] \subseteq (\alpha - \delta, \alpha + \delta) = I$. Hence f is bounded on $[a_n, b_n]$, which contradicts the choice of $[a_n, b_n]$.

(ii) Let $\alpha = b$.

Then clearly, $\alpha = b_n = b$ for all n. Then since f is continuous at α, for $\epsilon = 1$, $\exists\, \delta > 0$ such that
$x \in I = (\alpha - \delta, \alpha] \Rightarrow |f(x) - f(\alpha)| < 1$.
So, as before, f is bounded on I, also there exists n such that $[a_n, b_n] \subseteq I$. Hence f is bounded on $[a_n, b_n]$, which is a contradiction.

(iii) Let $a = \alpha$.

Then clearly, $a = a_n = \alpha$ for all n. Then since f is continuous at α, for $\epsilon = 1$, $\exists\, \delta > 0$ such that
$x \in I = [\alpha, \alpha + \delta) \Rightarrow |f(x) - f(\alpha)| < 1$.
Hence, as before, f is bounded on I. Choose n such that $b_n - a_n = \frac{b-a}{2^n} < \delta$. Then $\alpha = a_n < b_n < a_n + \delta = \alpha + \delta$. Hence $[a_n, b_n] \subseteq [\alpha, \alpha + \delta) = I$. Hence f is bounded on $[a_n, b_n]$, which is a contradiction.

Hence our assumption gives a contradiction in all cases. Hence f is bounded on $[a, b]$.

Example: The function $f(x) = x^2 + 1$ is continuous and bounded on $(0, 1)$ but does not attain infimum or supremum on $(0, 1)$.
Here supremum of $f = 2$ and infimum $f = 1$. If there is $c \in (0, 1)$ such that $f(c) = 2$, then $c^2 + 1 = 2$ that is, $c^2 = 1$, therefore $c = \pm 1$ which is not in $(0, 1)$. A contradiction to $c \in (0, 1)$. Therefore f does not attains its supre-

mum. Similarly f does not attain its infimum.

Example: The function $f(x) = \sin(\frac{1}{x}),\ x \neq 0$ and $f(0) = 0$ is discontinuous at $0 \in [-1, 1]$. Also it attains its infimum -1 and supremum 1 on $[-1, 1]$.

Example: Let $f(x) = 2x,\ \forall\ x \in [0, 1);\ \ f(x) = -x + 2,\ \forall\ x \in [1, 2)$ and $f(x) = 1,\ \forall\ \in [2, 3]$ be a function on $[0, 3]$.
The function f is discontinuous at points $1, 2 \in [0, 3]$, bounded on $[0, 3]$ and it does not attain its supremum 2 on $[0, 3]$.

Example: If every continuous function on a non-empty interval I is bounded then show that I is a closed bounded interval.
Solution: If I is unbounded then the function $f(x) = x$ is continuous and unbounded on I. Therefore I must be bounded.
Since I is bounded in $\mathbb{R}$, by completeness property of $\mathbb{R}$, $a = \inf(I)$ and $b = \sup(I)$ exists in $\mathbb{R}$.
If I is not closed then either a or b does not belong to I. Without loss of generality, suppose $a \notin I$. Therefore the function $g(x) = \frac{1}{x-a}$ is continuous and unbounded on I,a contradiction.
Therefore I must be closed.

Theorem: Extreme - Value Theorem For Continuous Functions.
If f is a continuous function on a closed and bounded interval $[a, b]$, then there exist points c and d in $[a, b]$ such that $f(c) = \sup f$ and $f(d) = \inf f$.

(Here $\sup f = \sup S$ and $\inf f = \inf S$, where $S = \{f(x) | x \in [a, b]\}$.)

Proof: We know that since f is a continuous function on the closed and bounded interval $[a, b]$, f is bounded on $[a, b]$. Hence $M = \sup S$ and $m = \inf S$ both exist on $[a, b]$.

We will prove that f attains its supremum in $[a, b]$. The result for the infimum will follow as a consequence because $\inf f = \sup(-f)$ and $-f$ is continuous on $[a, b]$ since f is so.

Let $M = \sup f$. If possible, suppose that there is no x in $[a, b]$ such that $f(x) = M$.
Let $g(x) = M - f(x)$. Then $g(x) > 0,\ \forall x \in [a, b]$. Also, g is continuous on $[a, b]$. Hence $\frac{1}{g}$ is continuous on $[a, b]$. Hence $\frac{1}{g}$ is bounded on $[a, b]$. So, $\exists\, K > 0$ such that $\forall\, x \in [a, b]$, $\frac{1}{g(x)} < K$,
i.e. $\frac{1}{M - f(x)} < K$ or $M - f(x) > \frac{1}{K}$.
$\therefore \forall\, x \in [a, b], f(x) < M - \frac{1}{K}$,

which is contradiction since M is the least upper bound (supremum) of f on $[a,b]$ and $M - \frac{1}{K} < M$.
$\therefore \exists\ c \in [a,b]$ such that $f(c) = \sup f$.

Theorem: Bolzano's Theorem (Location of roots theorem).
Let f be a continuous function on a closed and bounded interval $[a,b]$ and assume that $f(a)$ and $f(b)$ have opposite sings. Then there is at least one c in the open interval (a,b) such that $f(c) = 0$.

Proof: Assume $f(a) < 0$ and $f(b) > 0$. Let $S = \{x \in [a,b] \mid f(x) \leq 0\}$. Then $a \in S$. Also, $S \subseteq [a,b]$ so that S is a non-empty bounded set of real numbers. Hence the supremum of S exists, say c.

Now the possibilities are: $f(c) < 0$ or $f(c) > 0$ or $f(c) = 0$.
Let $f(c) < 0$. As f is continuous at $c \in [a,b]$, $\lim\limits_{x \to c} f(x) = f(c)$. Hence taking $\epsilon = \frac{-1}{2} f(c)$, $\exists\, \delta > 0$ such that
$x \in (c-\delta, c+\delta) \Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$,
so $x \in [c, c+\delta) \Rightarrow f(x) < \frac{1}{2} f(c) < 0$.
Hence $[c, c+\delta) \subseteq S$. In particular, $c + \frac{1}{2}\delta \in S$. This is a contradiction because $c = \sup S$ and $c + \frac{1}{2}\delta > c$. Hence $f(c) < 0$ is not possible.
Let $f(c) > 0$. As f is continuous at $c \in [a,b]$, $\lim\limits_{x \to c} f(x) = f(c)$. Hence taking $\epsilon = \frac{1}{2} f(c)$, $\exists\, \delta > 0$ such that
$x \in (c-\delta, c+\delta) \Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$,
so $x \in (c-\delta, c) \Rightarrow 0 < \frac{1}{2} f(c) < f(x)$.

But $c = \sup S$. Hence corresponding to the above δ, there exists a point $x_0 \in S$ such that $c > x_0 > c - \delta$. Then $f(x_0) \leq 0$ as x_0 is in S. This is a contradiction since as shown above, $f(x) > 0$ in $(c-\delta, c)$.

Hence we must have $f(c) = 0$, as required.

Note: The above theorem says that if f is a continuous function on a closed and bounded interval $[a,b]$ with $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then graph of the function f intersects X axis at a point between a and b.

Theorem: The Intermediate - Value Theorem for Continuous Functions.
Let f be a continuous function on a closed and bounded interval $[a,b]$. Let $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$ and $f(x_1) \neq f(x_2)$. Then f takes on every value between $f(x_1)$ and $f(x_2)$ somewhere in the interval (x_1, x_2).

Proof: Since $f(x_1) \neq f(x_2)$, suppose $f(x_1) < f(x_2)$ and let K be such that $f(x_1) < K < f(x_2)$.

Let $g(x) = f(x) - K,\ x \in [x_1, x_2]$.
As f is continuous on $[x_1, x_2] \subseteq [a, b]$, g is continuous on $[x_1, x_2]$. Note that

$$\begin{aligned} g(x_1) &= f(x_1) - K < 0 \qquad \text{and} \\ g(x_2) &= f(x_2) - K > 0. \end{aligned}$$

$\therefore$ by Bozano's Theorem $\exists\ c \in (x_1, x_2)$ such that $g(c) = 0$ i.e. $f(c) - K = 0$ i.e. $f(c) = K$.

Note: Converse of the intermediate value theorem is not true.
For example the function $f(x) = \sin(\frac{1}{x}),\ x \neq 0$; $f(0) = 0$ satisfies the intermediate value property on $[-1, 1]$, but f is not continuous at $0 \in [-1, 1]$.

Example: Given a real-valued function f which is continuous on the closed interval $[0, 1]$. Assume that $0 \leq f(x) \leq 1$ for each $x \in [0, 1]$. Prove that there is at least one point c in $[0, 1]$ for which $f(c) = c$. Such a point is called a fixed point of f.

Solution: Let $g(x) = f(x) - x,\ x \in [0, 1]$. Then g is continuous in $[0, 1]$ and

$$\begin{aligned} & g(0) = f(0) - 0 = f(0) \geq 0, \\ \text{and} \quad & g(1) = f(1) - 1 \leq 0. \end{aligned}$$

$\therefore$ by Bolzano's theorem $\exists\, c \in [0, 1]$ such that $g(c) = 0$ i.e. $f(c) - c = 0$ i.e. $f(c) = c$.

Example: Let f be a continuous function on a closed and bounded interval $[a, b]$. Show that the image of $[a, b]$ under f, namely

$$f([a, b]) = \{f(x) \mid x \in [a, b]\},$$

is a closed and bounded interval.

Solution: By Boundedness theorem, the supremum and infimum of $f([a, b])$ both exist.
Let $m = \inf f([a, b]), M = \sup f([a, b])$.
By extreme value theorem, $\exists\ c, d \in [a, b]$ such that $m = f(c)$ and $M = f(d)$.
Hence by intermediate value theorem, $f([a, b]) = [m, M] = [f(c), f(d)]$, as was to be shown.

Example: Show that the equation $2^x x - 1 = 0$ has at least one root in $(0, 1)$.

Solution: Let $f(x) = 2^x x - 1$. Clearly, f is continuous in $[0, 1]$.
Since $f(0) = 2^0 \times 0 - 1 = -1 < 0$

and $f(1) = 2^1 \times 1 - 1 = 2 - 1 = 1 > 0$,
by Bolzano's theorem it follows that $\exists \quad c \in (0,1)$ such that $f(c) = 0$ i.e. $2^c \times c - 1 = 0$.
$\therefore c \in (0,1)$ is a root of $2^x \times x - 1 = 0$.

Exercises:

1. Show that the equation $x = \cos x$ has a solution in the interval $\left[0, \frac{\pi}{2}\right]$.
2. Show that the polynomial $p(x) = x^4 + 7x^3 - 9$ has at least two real roots.
3. Does there exist a function which is bounded on $[a, b]$ but does not attain maximum or minimum on $[a, b]$? Justify your answer.
4. Let f be a polynomial of degree n, say $f(x) = \sum_{k=0}^{n} a_k x^k$, such that the first and last coefficients a_0 and a_n have opposite signs. Show that $f(x) = 0$ for at least one positive x.
5. Let $f(x) = \tan x$. Although $f\left(\frac{\pi}{4}\right) = 1$ and $f\left(\frac{3\pi}{4}\right) = -1$, there is no x in the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ such that $f(x) = 0$. Explain why this does not contradict Bolzano's theorem.

Exercises on Limits and Continuity

1. Prove the following by using the definition of limit:

   (a) $\lim_{x \to 1} (x^2 + 9) = 10$ (b) $\lim_{x \to 3} \frac{2}{x+9} = \frac{1}{6}$

   (c) $\lim_{x \to -6} \frac{x+4}{x-2} = \frac{1}{4}$ (d) $\lim_{x \to 1} \frac{x+3}{3+\sqrt{x}} = 1$.

2. Compute the following limits:

   (a) $\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$ (b) $\lim_{x \to 0} \frac{x^3 - 7x^4}{x^3}$

   (c) $\lim_{x \to 2} \frac{1}{x^2}$ (d) $\lim_{x \to a} \frac{x^2 - a^2}{x^2 + 2ax + a^2}, a \neq 0$.

3. Use $\lim_{x \to 0} \frac{\sin x}{x} = 1$ to evaluate the following limits:

   (a) $\lim_{x \to 0} \frac{\sin 2x}{x}$ (b) $\lim_{x \to 0} \frac{\sin 5x - \sin 3x}{x}$

(c) $\lim_{x\to 0} \dfrac{\sin x - \sin a}{x-a}$ (d) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$.

4. Compute following limits using sandwich theorem:

(a) $\lim_{x\to\infty} \dfrac{5-\cos x}{x+9}$ (b) $\lim_{x\to 0^-} x^3 \cos\left(\dfrac{3}{x}\right)$

(c) $\lim_{x\to -\infty} \dfrac{3x^2-\sin 4x}{x^2+10}$ (d) $\lim_{x\to 1} \dfrac{x^2}{x^2+1}$.

5. Compute following limits:

(a) $\lim_{x\to\infty} \dfrac{x^6-10}{x^6+10}$ (b) $\lim_{x\to -\infty} \dfrac{e^x}{2+3e^{3x}}$

(c) $\lim_{x\to\infty} \dfrac{7^x}{5^x+3^x}$.

6. Find the number A which makes the function

$f(x) = \begin{cases} x^2-2 & , \quad x<1 \\ Ax-4 & , \quad 1\le x \end{cases}$ continuous at $x=1$.

7. Discuss the continuity of the function f defined on $\mathbb{R}$ thus:

$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$

8. Discuss the continuity of

$f(x) = \begin{cases} \frac{x-6}{x-3} & , \quad x<0 \\ 2 & , \quad x=0 \\ \sqrt{x^2+4} & , \quad x>0. \end{cases}$

9. Discuss the continuity of $f(x) = \frac{x^3+1}{x^2+1}$ at $x=-1$.

10. Determine the set of points at which $f(x) = \frac{x^2+3x+5}{x^2+3x-7}$ is continuous.

11. Determine the points at which $f(x) = \sqrt{x^2-2x}$ is continuous.

12. Examine the continuity of the following functions at $x=0$:

(a) $f(x) = \begin{cases} (1+x)^{1/x} & , \quad x\ne 0 \\ 1 & , \quad x=0 \end{cases}$

(b) $f(x) = \begin{cases} (1+2x)^{1/x} & , \quad x\ne 0 \\ e^2 & , \quad x=0 \end{cases}$

(c) $f(x) = \begin{cases} e^{-1/x^2} & , \quad x \neq 0 \\ 1 & , \quad x = 0. \end{cases}$

(d) $f(x) = \begin{cases} \frac{e^{x^2}}{e^{1/x^2}-1} & , \quad x \neq 0 \\ 1 & , \quad x = 0. \end{cases}$

13. Let
$f(x) = \begin{cases} \sin x & , \quad x \leq c \\ ax + b & , \quad x > c \end{cases}$
where a, b and c are constants. If b and c are given, find all values of a (if any exist) for which f is continuous at the point $x = c$.

14. (a) Use the inequality $|\sin x| < |x|$, for $0 < |x| < \frac{\pi}{2}$, to prove that the sine function is continuous at 0.
(b) Use part (a) to prove that the cosine function is continuous at 0.
(c) Use parts (a) and (b) to prove that the sine and cosine functions are continuous at any real x.

15. Use the inequality $0 < \cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$ for $0 < x < \frac{\pi}{2}$, to prove that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

16. Given a real-valued function f which is continuous on a closed and bounded interval $[a, b]$. Assume that $f(a) \leq a$ and that $f(b) \geq b$. Prove that f has a fixed point in $[a, b]$.

17. Show that every polynomial of odd degree with real coefficients has at least one real root.

18. Let f be a continuous function on interval $[0, 1]$ such that
$f(0) = f(1)$. Show that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.
(Hint: Consider $g(x) = f(x) - f(x + \frac{1}{2}), \quad x \in [0, 1/2]$).

19. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and takes on only rational values. Show that f is a constant function.

CHAPTER 3
Differentiation

The derivative is a central concept of differential calculus. The deriative is a measure of how a function changes as its input changes. It can be thought as how much one quantity is changing in response to changes in some other quantity. The process of finding the derivative is called differentiation.

The Derivative:

Definition: Suppose a function f is defined on an open interval containing a point a. If the limit, $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists, then we say that f is differentiable (or derivable) at a and the derivative of f at a is denoted by $f'(a)$.

So,
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

If we take $x = a + h$, then
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Right hand and left hand derivatives:
If $\lim\limits_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$ exists, then it is called the right hand derivative of f at a, and is denoted by $f'_+(a)$. Similarly, if $\lim\limits_{h \to 0^-} \dfrac{f(a+h) - f(a)}{h}$ exists, then it is called the left hand derivative of f at a, and is denoted by $f'_-(a)$. Clearly, $f'(a)$ exists iff $f'_+(a)$ and $f'_-(a)$ both exist and are equal.

Examples:

1. The derivative of a constant function is 0 (zero) at any real number a.

2. Let $f(x) = x^2$.
   To find the derivative of $f(x)$ at point a.
   Consider $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

$$= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h}$$
$$= \lim_{h\to 0} 2a + h$$
$$= 2a.$$

Therefore, $f'(a) = 2a$.

3. Let $f(x) = \sqrt{x},\ x \geq 0$. Let $ageq0$ be any point.
   Then $\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$
   $$= \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$
   $$= \lim_{x\to a} \frac{1}{\sqrt{x} + \sqrt{a}}.$$
   Hence, if $a \neq 0$, then $f'(a) = \frac{1}{2\sqrt{a}}$ and
   if $a > 0$, then $f'_+(0) = \lim_{x\to 0^+} \frac{1}{\sqrt{x}}$ does not exist so that f is not derivable at 0.

4. Let $f(x) = |x|,\ \forall x \in \mathbb{R}$.
   Then $\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$
   $$= \lim_{x\to a} \frac{|x| - |a|}{x - a}$$
   $$= \lim_{x\to a} \frac{x^2 - a^2}{(x-a)(|x| + |a|)}$$
   Hence, if $a \neq 0$, then $f'(a) = \frac{a}{|a|} = \frac{a}{a} = 1 \quad \text{if } a > 0$
   $$= \tfrac{a}{-a} = -1 \ \text{ if } a < 0.$$
   But if $a = 0$, then
   $f'_+(0) = \lim_{x\to 0^+} \frac{x}{|x|} = 1$ and $f'_-(0) = \lim_{x\to 0^-} \frac{x}{|x|} = -1$.
   So, $f'_+(0) \neq f'_-(0)$.
   Hence, f is not derivable at 0.

Theorem 1 *If f is derivable at a point a, then f is continuous at a.*

Proof: Suppose f is derivable at a, so that $\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ exists and is $f'(a)$. Hence

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)+f(a)\right]\\ &= f'(a)\cdot 0 + f(a)\\ &= f(a).\end{aligned}$$

Therefore f is continuous at a.

Note: The converse of the above theorem is not true.
For example, let $f(x) = |x|, \forall x \in \mathbb{R}$.
Then f is continuous at 0, but as shown in example 4 above, f is not derivable at 0.

Example: Determine whether the function $h(x) = x|x|, x \in \mathbb{R}$, is differentiable and find the derivative if it exists.
Solution: Here,
$$\begin{aligned}h(x) = x|x| &= x^2 && \text{if } x > 0\\ &= 0 && \text{if } x = 0\\ &= -x^2 && \text{if } x < 0.\end{aligned}$$

Clearly,
$$\begin{aligned}h'(x) &= 2x && \text{if } x > 0\\ &= -2x && \text{if } x < 0.\end{aligned}$$

Let us check whether h is differentiable at 0.

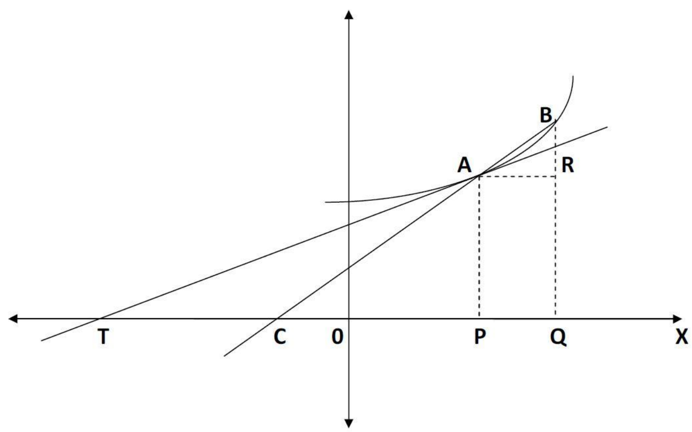
$$\begin{aligned}h'_+(0) &= \lim_{x\to 0^+}\frac{h(x)-h(0)}{x-0}\\ &= \lim_{x\to 0}\frac{x^2}{x}\\ &= \lim_{x\to 0} x\\ &= 0,\end{aligned}$$

$$\begin{aligned}h'_-(0) &= \lim_{x\to 0^-}\frac{h(x)-h(0)}{x-0}\\ &= \lim_{x\to 0}\frac{-x^2}{x}\\ &= \lim_{x\to 0} -x\\ &= 0.\end{aligned}$$

Therefore, $h'_+(0) = h'_-(0) = 0$.
Therefore, h is differentiable at 0 and $h'(0) = 0$.
Hence, h is differentiable at each $x \in \mathbb{R}$.

Geometrical interpretation of derivative:
Consider two points $A[a, f(a)]$ and $B[b, f(b)]$ on the graph of the curve $y =$

$f(x)$. Let the chord BA meet $X-$ axis at C so that $\angle XCB$ is its inclination. Draw ordinates AP, BQ and draw $AR \perp QB$.



We have $AR = PQ = h$ and $RB = QB - PA = f(a+h) - f(a)$.
Therefore, $\tan(\angle XCB) = \tan(\angle RAB) = \dfrac{RB}{AR} = \dfrac{f(a+h) - f(a)}{h}$. (1)

As h approaches 0, the point B moving along the curve approaches the point A, the chord AB approaches tangent line TA and $\angle XCB$ approaches $\angle XTA$ which is the inclination, say θ, of the tangent at A.
On taking limits as $h \to 0$, the equation (1) gives

$$\tan\theta = f'(a).$$

Thus, $f'(a)$ is the slope of the tangent to the curve $y = f(x)$ at the point $A[a, f(a)]$.

Note: The slope of the tangent at a point of a curve is also known as the Gradient of the curve at that point.

Example: Check whether the function $f(x) = |\log(x)|$ is differentiable at $x = 1$.

Solution: At $x = 1$,

$$\begin{aligned} f(x+h) - f(x) &= f(1+h) - f(1) \\ &= |\log(1+h)| - |\log 1| \\ &= |\log(1+h)| \end{aligned}$$

Therefore,
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\log(1+h)}{h} \qquad \text{if } h > 0 \\ &= \frac{-\log(1+h)}{h} \qquad \text{if } -1 < h < 0. \end{aligned}$$

So,
$$\begin{aligned} f'_+(1) &= \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{\log(1+h)}{h} \\ &= 1, \end{aligned}$$

and
$$\begin{aligned} f'_-(1) &= \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{-\log(1+h)}{h} \\ &= -1. \end{aligned}$$

Therefore, $f'_+(1) \neq f'_-(1)$.
So, $f'(1)$ does not exist.
Therefore, f is not differentiable at $x = 1$.

Example: If $f(x) = x \sin\left(\frac{1}{x}\right)$ when $x \neq 0$ and $f(0) = 0$, show that f is continuous but not derivable for $x = 0$.
Solution: We have

$$|f(x) - f(0)| = |x \sin\left(\frac{1}{x}\right)| = |x||\sin\left(\frac{1}{x}\right)| \leq |x|.$$

Given any $\epsilon > 0$ taking $\delta = \epsilon$, we have $|f(x) - f(0)| \leq |x| < \epsilon$ whenever $|x - 0| < \delta$.
Therefore, $\lim_{x\to 0} f(x) = 0 = f(0)$.
Therefore, f is continuous at 0. Next,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} \sin\left(\frac{1}{x}\right).$$

But, $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.
Therefore, $f'(0)$ does not exist.
Therefore, f is not derivable at $x = 0$.

Exercises:

1. Using the definition of derivative, find the derivative of the following functions:

   (a) $f(x) = \frac{1}{x}, \forall x \in \mathbb{R}, x \neq 0$

   (b) $f(x) = x^3$

   (c) $f(x) = x^n, n \in \mathbb{N}$

   (d) $f(x) = 3^x$

   (e) $f(x) = \cos x$.

2. The motion of a particle moving in a straight line is specified by the formula $S = t^2 + 2t + 3$. Find the velocity of motion (i) initially, (ii) at the end of 3 seconds.

3. Discuss the derivability of the function f given by
$$\begin{aligned} f(x) &= x^2, && x < 1 \\ &= 2 - x, && 1 \leq x \leq 2 \\ &= -x^2 + 3x - 2, && x > 2, \end{aligned}$$
   at $x = 1, 2$.

4. Give an example to show that the derivative of a continuous function is not always a continuous function.

5. Check whether the function f defined by $f(x) = |x-2| + |x|$ is derivable at $x = 0, 2$.

6. If a function f is defined by $$\begin{aligned} f(x) &= \frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, && x \neq 0 \\ &= 0, && x = 0, \end{aligned}$$
   show that f is continuous but not derivable at $x = 0$.

7. Show that the function $f(x) = |x-a|\phi(x)$ where $\phi(x)$ is a continuous function and $\phi(a) \neq 0$, has no derivative at the point $x = a$.

8. Find the coefficients c and d so that the function
   $f(x) = x^2, \qquad \text{if } x \le a$
   $\quad\;\; = cx + d, \qquad \text{if } x > a$
   is continuous and derivable at a.

Differentiability of a function over an interval:
A function f is said to be differentiable on an open interval I if it is differentiable at each $a \in I$. If $I = [c, d]$ then f is differentiable on I if
(i) $f'_+(c)$ exists and
(ii) $f'_-(d)$ exists and
(iii) $f'_+(a) = f'_-(a)$ for each a such that $c < a < d$.

Example: Define $f : [-1, 2] \to \mathbb{R}$ as follows:
$f(x) = x \qquad \text{if } -1 \le x < 0$
$\quad\;\; = \sin x \qquad \text{if } 0 \le x < 1$
$\quad\;\; = x^2 \qquad \text{if } 1 \le x \le 2.$

Solution: $f'(-1) = f'_+(-1) = \lim\limits_{x\to -1^+} \dfrac{f(x) - f(-1)}{x - (-1)} = \lim\limits_{x\to -1} \dfrac{x+1}{x+1} = 1.$

Also, $f'(2) = f'_-(2) = \lim\limits_{x\to 2^-} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x\to 2} \dfrac{x^2 - 4}{x - 2} = 4.$

At $x = 0$, $f'_+(0) = \lim\limits_{x\to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{\sin x - \sin 0}{x - 0} = 1$

and $f'_-(0) = \lim\limits_{x\to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{x - \sin 0}{x - 0} = 1.$

Therefore, $f'_+(0) = f'_-(0) = 1 \Rightarrow f'(0) = 1.$
At $x = 1$, $f'_+(1) = \lim\limits_{x\to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 1} = 2$

and $f'_-(1) = \lim\limits_{x\to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1} \dfrac{\sin x - 1}{x - 1}$ which does not exist $\Rightarrow$ $f'(1)$ does not exist.
Therefore, f is differentiable on $[-1, 1) \cup (1, 2]$.

Basic Rules of Differentiation:
There are a number of basic properties of derivatives that are very useful in

the calculation of derivatives of various combinations of functions. We now give a brief statement of some of these properties, which will be familiar to the reader from earlier courses.

Theorem 2 *Let f, g be functions defined on an interval containing point a and f, g be differentiable at a. Then*

1. *If $c \in \mathbb{R}$, then the function (cf) is differentiable at a and* $(cf)'(a) = cf'(a)$.

2. *The function $f + g$ is differentiable at a and* $(f + g)'(a) = f'(a) + g'(a)$.

3. *Product Rule:* *The function $f \cdot g$ is differentiable at a and* $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

4. *Quotient Rule:* *If $g(a) \neq 0$, then the function f/g is differentiable at a and* $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$.

Theorem 3 ***Chain Rule:***
Suppose I, J are open intervals and $f : I \to \mathbb{R}$, $f(I) \subset J$ and $g : J \to \mathbb{R}$. Suppose f is differntiable at $a \in I$ and g is differentiable at $f(a)$. Then the composite function $(g \circ f)$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Derivative of an inverse function: Suppose (i) f is a derivable function on a closed and bounded interval $[a, b]$ with $a < b$. Suppose (ii) $f'(x) \neq 0 \;\; \forall\, x \in [a, b]$.

Now, (i) implies that f is continuous on $[a, b]$ and so the range of f is also a closed and bounded interval, say $[c, d]$. Further, by (ii), it can be shown that f is actually either strictly increasing or strictly decreasing on $[a, b]$. Hence f is, in particular, a one-one function on $[a, b]$. Hence the inverse function, f^{-1}, of f exists and its domain is $[c, d]$. Also, to describe the relation between f and f^{-1}, conveniently, let us write $g = f^{-1}$. Then for every x in $[a, b]$, $y = f(x)$

is a *unique* number in $[c, d]$. Conversely, for every y in $[c, d]$, there is a *unique* number x in $[a, b]$ such that $y = f(x)$ i.e. such that $x = g(y)$.

Thus for x in $[a, b]$ and y in $[c, d]$, we have

$$y = f(x) \quad \Leftrightarrow \quad x = g(y).$$

Now it can be shown that, under the conditions (i) and (ii) above,
(a) g is also continuous on $[c, d]$,
(b) the inverse function, g of f is differentiable on $[c, d]$ and for every $y \in [c, d]$,

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}, \quad \text{where} \;\; y = f(x),$$

$$\text{i.e.} \qquad g'(y) = \frac{1}{f'(x)}, \quad \text{where} \;\; y = f(x). \tag{$*$}$$

Examples:

1. Every polynomial function $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is differentiable at any $x \in \mathbb{R}$.

2. Let $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, $\forall x \in \mathbb{R}$ for which $\cos x \neq 0$. Then
$$\begin{aligned} f'(x) &= \frac{\cos x \frac{d}{dx} \sin x - \sin x \frac{d}{dx} \cos x}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x. \end{aligned}$$

3. Let $f = \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^5 + 4x + 3$ and $g = f^{-1}$. Find $g'(8)$.
Solution: f is continuous and one-one function on $\mathbb{R}$ and $f'(x) = 5x^4 + 4 \neq 0$ for any $x \in \mathbb{R}$. Therefore, the inverse function $g = f^{-1}$ is differentiable at every $x \in \mathbb{R}$.
Let $x_0 = 1 \Rightarrow f(x_0) = f(1) = 8$.
Thererfore, $g'(8) = \dfrac{1}{f'(1)} = 9$.

4. Let $f : (-\pi/2, \pi/2) \to (-1, 1)$ be defined as $f(x) = \sin x$.
Clearly, $f(x)$ is continuous and one-one function on $(-\pi/2, \pi/2)$ and

$f'(x) = \cos x \neq 0,\ \forall x \in (-\pi/2, \pi/2)$.
So, the inverse function $g(y) = f^{-1}(y) = \sin^{-1} y$ is differentiable at every $y \in (-1, 1)$ and
$g'(y) = \dfrac{1}{f'(x)} = \dfrac{1}{\cos x} = \dfrac{1}{\sqrt{1-\sin^2 x}} = \dfrac{1}{\sqrt{1-y^2}},\ y \in (-1, 1)$.

5. Find the derivatives of the following functions:

   (a) $f(x) = (2x^2 + 5x + 3)^4$
   (b) $f(x) = 3^{\sin x}$
   (c) $\sqrt{x + \sqrt{x + \sqrt{x}}}$
   (d) $f(x) = \sinh 3x\ \cosh(x/5)$

   Solution:

   (a) $f'(x) = 4(2x^2 + 5x + 3)^3(4x + 5)$
   (b) $f'(x) = 3^{\sin x} \log 3(\cos x) = 3^{\sin x} \cos x \log 3$
   (c) $f'(x) = \dfrac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \left[1 + \dfrac{1}{2\sqrt{x + \sqrt{x}}}\left(1 + \dfrac{1}{2\sqrt{x}}\right)\right]$
   (d) $f'(x) = 3\cosh 3x\ \cosh(x/5) + \dfrac{1}{5}\sinh 3x\ \sinh(x/5)$.

6. Show that the function $y = xe^{-x}$ satisfies the equation $xy' = (1 - x)y$.
   Solution: $y' = e^{-x} - xe^{-x} = (1 - x)e^{-x}$.
   Therefore, $xy' = (1 - x)y$.

Exercises:

1. Find the derivatives of the following functions:

   (a) $y = (\sin x)^{\cos x}$
   (b) $y = \sqrt[3]{\dfrac{\sin 2x}{1 - \sin 2x}}$
   (c) $f(x) = \coth(\tan x) - \tanh(\cot x)$
   (d) $f(x) = \dfrac{e^x + \sin x}{xe^x}$

(e) $f(x) = |\sin x|$

(f) $f(x) = \dfrac{x}{1+x^2}$.

2. Determine whether each of the following functions is differentiable and find the derivative wherever it exists:

   (a) $f(x) = x + |x|$

   (b) $f(x) = |x| + |x+1|$

   (c) $f(x) = |\sin x|$

   (d) $f(x) = \sqrt{1-\sqrt{1-x^2}}$.

3. Show that the function $y = \dfrac{-e^{-x^2}}{2x^2}$ satisfies the differential equation $xy' + 2y = e^{-x^2}$.

4. Given that the function $h(x) = x^3 + 2x + 1$ for $x \in \mathbb{R}$, has an inverse h^{-1} on $\mathbb{R}$, find the value of $h^{-1\prime}(y)$ at the points corresponding to $x = 0, 1, -1$.

5. Prove that the derivative of a differentiable even function is an odd function and the derivative of an odd function is an even function.

6. Prove that the derivative of a periodic function with period T is a periodic function with period T.

7. Find $f'(x)$ if $f(x) = \begin{vmatrix} x & 1 & 0 \\ x^2 & 2x & 2 \\ x^3 & 3x^2 & 6x \end{vmatrix}$.

8. For $u = \dfrac{1}{2}\log\left(\dfrac{1+v}{1-v}\right)$. Check the relation $\dfrac{du}{dv}\dfrac{dv}{du} = 1$.

Mean Value Theorems:

In this section we are going to study the mean value theorems, which relate the values of a function to values of its derivative.

We begin by studing the relationship between the relative extrema of a function and the values of its derivative.

Let I be an interval. A function $f : I \to \mathbb{R}$ is said to have:

1. a relative maximum at point $a \in I$ if there exists a neighborhood V of a such that $f(x) \leq f(a)$ for all $x \in V \cap I$.

2. a relative minimum at point $a \in I$ if there exists a neighborhood V of a such that $f(a) \leq f(x)$ for all $x \in V \cap I$.

We say, f has a relative exremum at $a \in I$ if it has either a relative maximum or a relative minimum at a.

Theorem 4 *: Let a be an interior point of I and $f : I \to \mathbb{R}$ have a relative extremum at a. If $f'(a)$ exists then $f'(a) = 0$.*

Proof: We prove the result for the case that f has a relative maximum at a. The proof for the case of a relative minimum is similar.
Now, $f'(a)$ exists and $f'(a) \in \mathbb{R}$.
So exactly one of the following is true:

1. $f'(a) > 0$,

2. $f'(a) < 0$,

3. $f'(a) = 0$

First, suppose $f'(a) > 0$.
By definition of derivative, there exists a neighborhood $V \subseteq I$ of a such that $\dfrac{f(x) - f(a)}{x - a} > 0$ for $x \in V,\ x \neq a$.
If $x \in V \subseteq I$ and $x > a$, then
$f(x) - f(a) = (x - a)\dfrac{f(x) - f(a)}{x - a} > 0$.
So, $f(x) > f(a)$ which contradicts to the hypothesis that$f$ has relative maximum at a.
Therefore $f'(a) > 0$ is impossible.
Similarly, we cannot have $f'(a) < 0$.
Therefore, we must have $f'(a) = 0$.

Note:

1. Let $f : I \to \mathbb{R}$ and $c \in I$. If $f'(c) = 0$ or $f'(c)$ is undefined ,then c is called as a critical point for f.

2. The only domain points where a function can assume extreme values are critical points and end points.

Theorem 5 *: **(Rolle's Theorem)***
Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is

1. continuous on $[a, b]$,

2. derivable on (a, b) and

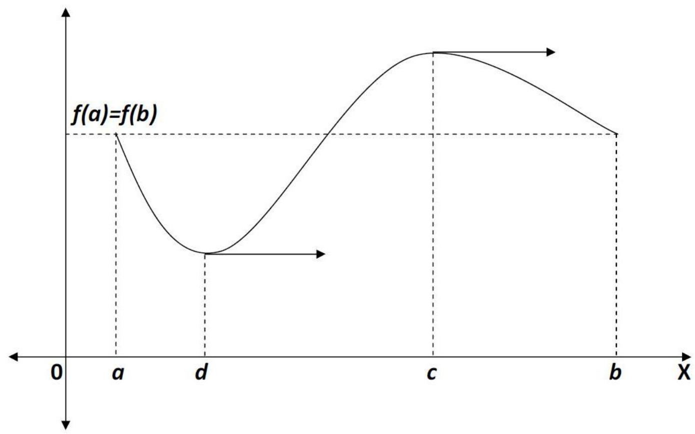
3. $f(a) = f(b)$.

Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof: f is continuous on $[a, b]$. So, f is bounded on $[a, b]$ and attains its maximum value M and minimum value m on $[a, b]$.
Hence, $\exists\ c, d \in [a, b]$ such that $f(c) = M$ and $f(d) = m$. Clearly, $m \leq M$.
If $m = M$, then f is constant on $[a, b]$. Therefore, $f'(x) = 0,\ \forall x \in (a, b)$.
If, $m < M$, then the numbers m and M cannot both be equal to the equal values $f(a)$ and $f(b)$.
So, first let $M = f(c) \neq f(a)$.
Then $c \in (a, b)$. Hence, by *(2)*, $f'(c)$ exists.
We now show that $f'(c) = 0$.
Since $f(x) \leq f(c),\ \forall x \in [a, b]$, we have
$$\frac{f(x) - f(c)}{x - c} \geq 0 \text{ for } x < c$$
and $\dfrac{f(x) - f(c)}{x - c} \leq 0$ for $x > c$.
Hence, as $x \to c^-$, we get $f'_-(c) \leq 0$
and as $x \to c^+$, we get $f'_+(c) \geq 0$.
Since $f'(c) = f'_-(c) = f'_+(c)$, we thereforeget $f'(c) = 0$.
So, c is the required point.
A similar proof shows that if $m = f(d) \neq f(a)$, then d is the required point.

Geometrical interpretation of Rolle's theorem:
Geometrically, Rolle's theorem says that, if f is continuous function on $[a, b]$ and if the graph of f has a tangent at each point between the points $(a, f(a))$ and $(b, f(b))$, which are on the same level, then there is a point c $(a < c < b)$ such that the tangent to the graph of f at point $(c, f(c))$ is parallel to the

$X-$axis, that is, f reaches its maximum or minimum value at an interior point of $[a, b]$.



Examples:

1. Let p be a polynomial of degree $n \geq 2$. Show that between any two distinct roots of p there is a root of p'.
   Solution: Let $p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$, $a_0 \neq 0$
   be a polynomial of degree n and let a and b be distinct roots of $p(x)$ with $a < b$.
   So, $p(a) = p(b) = 0$. Clearly, p is continuous on $[a, b]$ and derivable on (a, b).
   All conditions of Rolle's theorem are satisfied.
   So, there exists $c \in (a, b)$ such that $p'(c) = 0$.
   Therefore, a root of p' lies between the roots a, b of p.

2. Verify Rolle's theorem for the function
   $f(x) = e^x(\sin x - \cos x)$ on $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$.
   Solution: Here, $f(x) = e^x(\sin x - \cos x)$.
   $f(\pi/4) = e^{\pi/4}(\sin(\pi/4) - \cos(\pi/4))$

$= e^{\pi/4}((1/\sqrt{2}) - 1/\sqrt{2})$
$= 0.$
$f(5\pi/4) = e^{5\pi/4}(\sin(5\pi/4) - \cos(5\pi/4))$
$= e^{\pi/4}((-1/\sqrt{2}) + 1/\sqrt{2})$
$= 0.$
Therefore, $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$.
Clearly, f is continuous on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. Also, f is differentiable and
$f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x)\ \forall x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.
So, all conditions of Rolle's theorem are satisfied.
Hence, there must exist $c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$.
To verify this, let $f'(c) = 0$.
i.e. $e^c(\sin c - \cos c) + e^c(\cos c + \sin c) = 0$
i.e. $2e^c \sin c = 0$
i.e. $\sin c = 0 \qquad (\because 2e^c \neq 0)$
i.e. $c = n\pi,\ n \in \mathbb{Z}$.
So, $c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ satisfies $f'(c) = 0$.
Hence, Rolle's theorem is verified.

3. If $f(x) = (x-3)\log x$ then show that the equation $x \log x = 3 - x$ for some $x \in (1,3)$.
Solution: Here, $f(x) = (x-3)\log x$.
$f(1) = (1-3)\log 1 = 0,\ f(3) = (3-3)\log 3 = 0.$
$\therefore f(1) = f(3).$
Clearly, f is continuous on $[1,3]$ and differentiable on $(1,3)$.
$\therefore$ By Rolle's theorem, $\exists c \in (1,3)$ such that $f'(c) = 0$.
Here, $f(x) = (x-3)\log x$.
Therefore, $f'(x) = (x-3)\frac{1}{x} + \log x,\ x > 0$.
$\therefore f'(c) = 0$ gives $\frac{c-3}{c} + \log c = 0$.
$\therefore (c-3) + c\log c = 0.$
$\therefore c\log c = 3 - c.$
$\therefore c \in (1,3)$ satisfies the equation $x \log x = 3 - x$.

Theorem 6 *: (Lagrange's Mean Value Theorem)*
Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is

1. *continuous on $[a, b]$,*
2. *derivable on (a, b).*

Then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

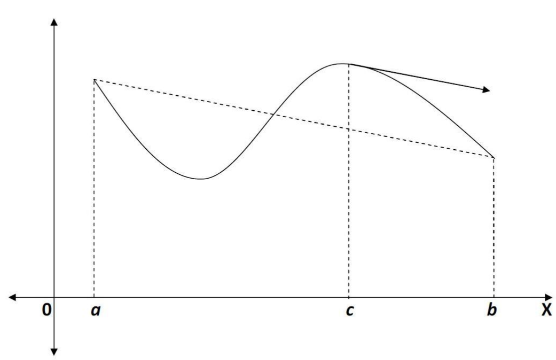
Proof: For $a \leq x \leq b$, we define

$$\phi(x) = f(x) - kx$$

where k is chosen such that $\phi(a) = \phi(b)$.
For this, we require $f(a) - ka = f(b) - kb$ or $k = \dfrac{f(b) - f(a)}{b - a}$. [I]
Now f is continuous on $[a, b]$ and differentiable on (a, b). So, ϕ is continuous on $[a, b]$ and differentiable on (a, b). Also, $\phi(a) = \phi(b)$.
Therefore, by Rolle's theorem there exists $c \in (a, b)$ such that $\phi'(c) = 0$ i.e. $f'(c) = k$ i.e.

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{using [I]}$$

Geometrical Interpretation of LMVT:



Geometrically, LMVT states that there exists a point $c \in (a,b)$, such that the tangent at point $(c, f(c))$ is parallel to the chord joining points $(a, f(a))$ and $(b, f(b))$.

Note:

1. The expression $\dfrac{f(b)-f(a)}{b-a}$ gives the 'mean value' of the slope of f on $[a,b]$ and LMVT says that f' must attain this mean value somewhere on (a,b).

2. If we write $b-a=h$ and $\dfrac{c-a}{h}=\theta$, then $0<\theta<1$ and $c=a+\theta h$. Hence, LMVT can be stated as: If $f=[a,a+h]\to\mathbb{R}$ is

   (a) continuous on $[a,a+h]$,

   (b) derivable on $(a,a+h)$,

   then there exists $\theta \in (0,1)$ such that

$$f(a+h)-f(a)=hf'(a+\theta h).$$

Examples:

1. Verify LMVT for$f(x)=\log x$ on $[1,e]$.
   Solution: Here $f(x)=\log x,\ x\in[1,e]$.
   Clearly, f is continuous on $[1,e]$ and differentiable on $(1,e)$. Also, $f'(x)=\dfrac{1}{x}$.
   To verify LMVT we have to find $c\in(1,e)$ such that $f'(c)=\dfrac{f(e)-f(1)}{e-1}$
   i.e. $\dfrac{1}{c}=\dfrac{\log(e)-\log(1)}{e-1}$
   i.e. $\dfrac{1}{c}=\dfrac{1}{e-1}$
   Therefore, $c=e-1\in(1,e)$ since $e=2.718\ldots$.
   Hence LMVT is verified.

2. If $0 < a < b < 1$ then prove that

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}.$$

Solution: Consider $f(x) = \sin^{-1} x, \quad x \in [a, b]$.
Clearly, f is continuous on $[a, b]$. Also,
$f'(x) = \dfrac{1}{\sqrt{1-x^2}}, \quad \forall x \in (a, b)$.
Therefore, f is derivable on (a, b).
So, by LMVT, $\exists c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$
i.e. $\dfrac{1}{\sqrt{1-c^2}} = \dfrac{\sin^{-1} b - \sin^{-1} a}{b-a}$ [I]
Here, $0 < a < c < b < 1$, $\therefore a^2 < c^2 < b^2 < 1$.
$\therefore -a^2 > -c^2 > -b^2 > -1$.
$\therefore \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$.
$\therefore \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$.
Using [I], we therefore get $\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$.
$\therefore \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$.

3. Prove that $|\sin x - \sin y| \le |x - y|, \quad \forall x, y \in \mathbb{R}$.
Solution: Let $x, y \in \mathbb{R}$ and $f(t) = \sin t, \ t \in [x, y]$.
Clearly, f is continuous on $[x, y]$.
Also f is derivable and $f'(t) = \cos t, \ t \in (x, y)$.
So, by LMVT $\exists c \in (x, y)$ such that $f'(c) = \dfrac{f(y) - f(x)}{y - x}$
i.e. $\cos c = \dfrac{\sin y - \sin x}{y - x}$
so that $|\cos c| = \dfrac{|\sin y - \sin x|}{|y - x|}$.
We know that $|\cos c| \le 1, \quad \therefore \frac{|\sin y - \sin x|}{|y-x|} \le 1$.
$\therefore |\sin x - \sin y| \le |x - y|, \quad \forall x, y \in \mathbb{R}$.

Recall that a function $f : I \to \mathbb{R}$ is said to be strictly increasing on an interval I if for any points $x_1, x_2 \in I$ such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Similarly, f is strictly decreasing on I if for any points $x_1, x_2 \in I$ such that $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

Theorem 7 *: Suppose $a < b$ and f is derivable on (a, b). Then*

1. $f'(x) \neq 0,\ \forall x \in (a, b) \Rightarrow f$ *is a one-one function on* (a, b).
2. $f'(x) = 0,\ \forall x \in (a, b) \Rightarrow f$ *is a constant function on* (a, b).
3. $f'(x) > 0,\ \forall x \in (a, b) \Rightarrow f$ *is a strictly increasing function on* (a, b).
4. $f'(x) < 0,\ \forall x \in (a, b) \Rightarrow f$ *is a strictly decreasing function on* (a, b).

Proof: Let $a < x < y < b$, then by LMVT $\exists c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$, [I]

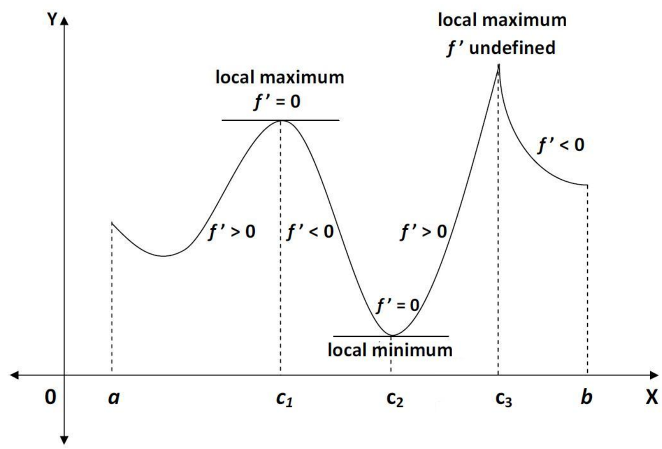
1. Suppose $f'(x) \neq 0\ \forall x \in (a, b)$.
   To show: f is a one-one function on (a, b).
   On the contrary, suppose f is not a one-one function on (a, b). Then there exist $x, y \in (a, b)$ such that $x < y$ and $f(x) = f(y)$. But then by [I], $\exists c \in (x, y)$ such that $f'(c) = 0$
   i.e. $\exists c \in (a, b)$ such that $f'(c) = 0$, which is a contradiction.
   So, f must be one-one on (a, b).

2. Suppose $f'(x) = 0\ \forall x \in (a, b)$.
   To show: f is a constant function on (a, b).
   On the contrary, suppose f is not constant on (a, b).
   Then $\exists x, y \in (a, b)$ such that $x < y$ and $f(x) \neq f(y)$, so by [I], we have,
   $\exists c \in (x, y)$ such that $f'(c) = \dfrac{f(y) - f(x)}{y - x} \neq 0$.
   Hence, $\exists c \in (a, b)$ such that $f'(c) = 0$ which is a contradiction.
   So, f must be constant on (a, b).

3. Suppose $f'(x) > 0,\ \forall x \in (a, b)$.
   Using [I], we have $f'(c) = \dfrac{f(y) - f(x)}{y - x} > 0$.
   Here, $x < y \Rightarrow y - x > 0$.
   So, $f(y) - f(x) > 0 \Rightarrow f(x) < f(y)$.
   So, for any $x, y \in (a, b)$ with $x < y$, we have $f(x) < f(y)$.
   Therefore, f is strictly increasing on (a, b).

4. Similarly, we can prove if $f'(x) < 0,\ \forall x \in (a,b)$, then f is strictly decreasing on (a,b).

Note: f is increasing on an interval I, if for any $x, y \in I$ with $x < y$, we have $f(x) \leq f(y)$.
Similarly, f is decreasing on I if for any $x, y \in I$ with $x < y$, we have $f(x) \geq f(y)$.
So, by above theorem, if $f'(x) \geq 0,\ \forall x \in I$, then f is increasing on I and if $f'(x) \leq 0,\ \forall x \in I$, then f is decreasing on I.



Thus if the derivative f' has a fixed sign on an interval, the f is monotonic on that interval. This fact leads us to a test, given below, for obtaining the local extreme points of f on $[a,b]$.

For example, in the above figure, consider values of f near the points c_1, c_2, c_3. Here f is continuous on $[a,b]$ and also derivable on $[a,b]$ *except* at c_3.

Now at points *to the left of* c_1, the tangent to the graph of f makes an *acute* angle with the positive $x-$axis, i.e. $f'(x) \geq 0$ at all these points. So f increases on the left of c_1. Also, at points *to the right of* c_1, the tangent to the graph of f makes an *obtuse* angle with the positive $x-$axis, i.e. $f'(x) \leq 0$ at all

these points. So f decreases on the right of c_1. Hence f is *locally maximum* at c_1. Similarly, the reader should check that f is *locally minimum* at c_2. Finally, since f is continuous at c_3 and derivable in a deleted neighbourhood of c_3, it can be similarly seen that f is *locally maximum* at c_3.
These observations lead to following test for existence of local extreme values.

Theorem 8 *: **(The First Derivative Test for Local Extreme Values)*** *Let f be continuous on an interval $I = [a, b]$ and let c be an interior point of I. Assume that f is differentiable on (a, c) and (c, b). Then*

1. *If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \geq 0$ for $c - \delta < x < c$ and $f'(x) \leq 0$ for $c < x < c + \delta$, then f has a local maximum at c.*

2. *If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \leq 0$ for $c - \delta < x < c$ and $f'(x) \geq 0$ for $c < x < c + \delta$, then f has a local minimum at c.*

Proof:

1. If $x \in (c - \delta, c)$ then by Lagrange's mean value theorem, there exists a point $c_x \in (x, c)$ such that

$$f(c) - f(x) = (c - x)f'(c_x)$$

   Since, $f'(c_x) \geq 0$, we have $f(c) - f(x) \geq 0$.
   $\therefore f(c) \geq f(x)$ for $x \in (c - \delta, c)$.
   Similarly, if $x \in (c, c + \delta)$ then by Lagrange's mean value theorem, there exists a point $c_x \in (c, x)$ such that

$$f(c) - f(x) = (c - x)f'(c_x)$$

   But here $f'(c_x) \leq 0$, also $(c - x) \leq 0$.
   $\therefore f(c) - f(x) \geq 0$
   $\therefore f(c) \geq f(x)$ for $x \in (c, c + \delta)$.
   Therefore, c is a point of local maximum for f.

2. Proof is similar.

Note: Suppose $I = [a, b]$ and $f : I \to \mathbb{R}$ is a derivable function. If c is a point of local extremum for f then c is either a critical point for f or an endpoint.

Examples:

1. Find the intervals on which $f(x) = -x^3 + 12x + 5,\ x \in (-3, 3)$, is increasing or decreasing. Where does the function assume extreme values and what are these values?
   Solution: The function f is continuous on $(-3, 3)$. Also, f' is defined for all $x \in (-3, 3)$ and
   $f'(x) = -3x^2 + 12 = -3(x^2 - 4)$
   $\quad = -3(x + 2)(x - 2)$.
   $\therefore f'(x) = 0$ for $x = 2, -2$.
   $\therefore x = -2, 2$ are critical points for f.
   These critical points subdivide $(-3, 3)$ into subintervals $(-3, -2),\ (-2, 2),\ (2, 3)$.

subintervals	(-3, -2)	(-2, 2)	(2, 3)
sign of $f'(x)$	$-$	$+$	$-$

   $\therefore f$ is increasing on $(-2, 2)$ and decreasing on $(-3, -2),\ (2, 3)$.
   So, by first derivative test, we conclude that $x = -2$ is a point of local minimum for f and $x = 2$ is a point of local maximum for f.
   $\therefore f(-2) = -11$ is minimum value for f and $f(2) = 21$ is maximum value for f.

2. Show that $x - \dfrac{x^2}{2} < \log(1 + x) < x - \dfrac{x^2}{2(1 + x)}$ for $x > 0$.
   Solution: Consider $f(x) = \log(1 + x) - \left(x - \dfrac{x^2}{2}\right),\ x \geq 0$.
   $\therefore f'(x) = \dfrac{1}{1 + x} - (1 - x) = \dfrac{x^2}{1 + x} > 0, \quad \forall x > 0.$
   So, $f(x)$ is strictly increasing on $(0, \infty) \Rightarrow f(0) < f(x), \quad \forall x > 0$.
   $\therefore \log(1 + 0) < \log(1 + x) - \left(x - \dfrac{x^2}{2}\right)$
   $\therefore 0 < \log(1 + x) - \left(x - \dfrac{x^2}{2}\right)$

$\therefore x - \dfrac{x^2}{2} < \log(1+x).$ [I]

Now consider $g(x) = x - \dfrac{x^2}{2(1+x)} - \log(1+x), \quad x \geq 0.$

$\therefore g'(x) = \dfrac{x^2}{2(1+x)^2} > 0, \quad \forall x > 0.$

$\therefore g(x)$ is strictly increasing on $(0, \infty)$.

$\therefore g(0) < g(x)$ for $0 < x$.

$\therefore 0 < x - \dfrac{x^2}{2(1+x)} - \log(1+x).$

$\therefore \log(1+x) < x - \dfrac{x^2}{2(1+x)}.$ [II]

Using [I] and [II], we get

$x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}$ for $x > 0$.

Theorem 9 *: **(Cauchy's Mean Value Theorem)***

Let $a < b$. Suppose the functions $f, g : [a,b] \to \mathbb{R}$ are

1. *continuous on $[a,b]$*
2. *derivable on (a,b) and*
3. *$g'(x) \neq 0 \ \ \forall x \in (a,b)$.*

Then there exists a point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof: If $g(a) = g(b)$, then g satisfies all conditions of Rolle's theorem, so there exists $c \in (a,b)$ such that $g'(c) = 0$. But it is given that $g'(x) \neq 0, \ \ \forall x \in (a,b)$.

$\therefore g(a) \neq g(b)$.

For $a \leq x \leq b$, we define $\phi(x) = f(x) - kg(x)$

where k is a constant chosen such that $\phi(a) = \phi(b)$

i.e. $f(a) - kg(a) = f(b) - kg(b)$

i.e. $k = \dfrac{f(b)-f(a)}{g(b)-g(a)}.$ [I]

Since, f, g are

1. continuous on $[a, b]$ and
2. derivable on (a, b),

ϕ is also

1. continuous on $[a, b]$ and
2. derivable on (a, b).

Also, $\phi(a) = \phi(b)$.
$\therefore \phi$ satisfies all conditions of Rolle's theorem.
Hence, there exists a point $c \in (a, b)$ such that $\phi'(c) = 0$.
$\therefore f'(c) - kg'(c) = 0$.
$\therefore \dfrac{f'(c)}{g'(c)} = k. \qquad (\because g'(c) \neq 0)$
$\therefore \dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)} \qquad$ using [I]

Examples:

1. Verify Cauchy's mean value theorem for the functions $f(x) = e^x$ and $g(x) = e^{-x}$ in $[0, 1]$. Show that c is the arithmetic mean of a and b.
   Solution: f, g are continuous on $[0, 1]$. Also, f, g are differentiable on $(0, 1)$. Further
   $g'(x) = -e^{-x} \neq 0 \ \ \forall x \in (0, 1)$.
   Thus, all conditions of CMVT are satisfied.
   So, there must exist $c \in (a, b)$ such that
   $$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$
   $\therefore \dfrac{e^c}{-e^{-c}} = \dfrac{e^b - e^a}{e^{-b} - e^{-a}}$.
   $-e^{2c} = -e^{a+b}$.
   $\therefore 2c = a + b$.
   $\therefore c = \dfrac{a+b}{2} \in (a, b)$.
   Hence, CMVT is verified and c is the arithmetic mean of a and b.

2. Find c of Cauchy's mean value theorem for the functions $\cos x$ and $\sin x$ in $[0, \pi/2]$.
   Solution: Let $f(x) = \cos x,\ g(x) = \sin x$.
   Then both f, g are continuous on $[0, \pi/2]$ and derivable on $(0, \pi/2)$.
   Also, $g'(x) = \cos x \neq 0, \quad \forall x \in (0, \pi/2)$.
   So all conditions of CMVT are satisfied.
   By Cauchy's mean value theorem there exists a real number $c \in (0, \pi/2)$ such that
   $$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \text{[I]}$$
   $\therefore \dfrac{\sin c}{-\cos c} = \dfrac{\cos(\pi/2) - \cos 0}{\sin(\pi/2) - \sin 0}$
   $\therefore -\tan c = -1$
   $\therefore \tan c = 1$
   $\therefore c = \pi/4 \in (0, \pi/2)$.
   So, $c = \pi/4$ is the point satisfying [I].

3. Verify CMVT for the functions $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x}$ on $[a, b]$, $a > 0$. Show that the point c of Cauchy's mean value theorem is the harmonic mean of a and b. **Solution:** Here $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x}$.
   Clearly, both f, g are continuous on $[a, b]$ and derivable on (a, b), where $a > 0$.
   $\therefore$ By CMVT, $\exists c \in (a, b)$ such that
   $$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$
   $$\therefore \frac{-2/c^3}{-1/c^2} = \frac{1/b^2 - 1/a^2}{1/b - 1/a}$$
   $$\therefore \frac{2c^2}{c^3} = \frac{1}{b} + \frac{1}{a}$$
   $$\therefore \frac{2}{c} = \frac{a+b}{ab}$$
   $$\therefore c = \frac{2ab}{a+b} \in (a, b).$$
   Hence, CMVT is verified and c is the harmonic mean of a and b.

4. Show that for $0 < x < \pi/2$,

   (a) $\sin x < x < \tan x$

   (b) $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$.

   Solution:

   (a) For $0 \leq x < \pi/2$, let $f(x) = x - \sin x,\ g(x) = \tan x - x$.
   Then for all $x \in (0, \pi/2),\ f'(x) = 1 - \cos x > 0$ and $g'(x) = \sec^2 x - 1 > 0$.
   So, f, g are strictly increasing on $[0, \pi/2)$.
   Hence, for $0 < x,\ f(0) < f(x)$ and $g(0) < g(x)$
   i.e. $0 < x - \sin x$ and $0 < \tan x - x$
   i.e. $\sin x < x$ and $x < \tan x$
   $\therefore \sin x < x < \tan x$.

   (b) For $0 < x \leq \pi/2$, let $F(x) = \dfrac{\sin x}{x}$ and $F(0) = 1$.
   Then F is continuous on $[0, \pi/2]$.
   Also, on $(0, \pi/2),\ F'(x) = \dfrac{x \cos x - \sin x}{x^2} < 0$ since $x < \tan x$ by (a).
   Hence, F is strictly decreasing on $[0, \pi/2)$ and so for $x \in (0, \pi/2)$, we have $F(\pi/2) < F(x) < F(0)$.
   Therefore, $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$.

Exercises:

1. Verify Rolle's theorem, if applicable, for the given function.

   (a) $f(x) = |x|,\ x \in [-1, 1]$

   (b) $f(x) = x(x-2)e^{-x},\ x \in [0, 2]$

   (c) $f(x) = 9x^3 - 4x,\ x \in [-2/3, 2/3]$

   (d) $f(x) = \cos^2 x,\ x \in [-\pi/4, \pi/4]$.

2. Verify LMVT if applicable, for the given function.

   (a) $f(x) = e^x,\ x \in [-1, 1]$

(b) $f(x) = 3x^2 - 5x + 1,\ x \in [2, 5]$
(c) $f(x) = \sin x + \cos x,\ x \in [0, 2\pi]$
(d) $f(x) = \sqrt{x^2 - 4},\ x \in [2, 3]$.

3. Verify CMVT if applicable, for the given function.

   (a) $f(x) = e^x,\ g(x) = \dfrac{x^2}{1 + x^2},\ x \in [-3, 3]$
   (b) $f(x) = x^2 + 2,\ g(x) = x^3 - 1,\ x \in [1, 2]$.

4. Show that $\dfrac{x}{1 + x} < \log(1 + x) < x,\ x > 0$.

5. Show that $\dfrac{b - a}{1 + b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b - a}{1 + a^2},\ \forall\ 0 < a < b$.

6. Verify Cauchy's mean value theorem for functions $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ in $[a, b],\ a > 0$. Show that the number c of CMVT is the geometric mean of a and b.

7. Separate the intervals in which the polynomial $2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.

8. If $\dfrac{a_0}{n + 1} + \dfrac{a_1}{n} + \ldots + \dfrac{a_{n-1}}{2} + \dfrac{a_n}{1} = 0$, show that the equation $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$ has a root in $(0, 1)$.

9. If $f(x) = x(x + 1)(x + 2)(x + 3)$, show that the equation $f'(x) = 0$ has three real roots.

10. Using $f(x) = (4 - x) \log x$, show that $x \log x = 4 - x$ for some $x \in (1, 4)$.

11. Find the points of local extrema, the intervals on which functions are increasing or decreasing for

    (a) $f(x) = x + \dfrac{1}{x},\ x \neq 0$
    (b) $f(x) = x^3,\ x \in [-1, 1]$
    (c) $f(x) = x^5 - 5x^4 + 5x^3 - 1,\ x \in \mathbb{R}$
    (d) $f(x) = xe^{-x},\ x \in (0, 2)$

(e) $f(x) = x^2 + 3x + 5, \ x \in \mathbb{R}$

(f) $f(x) = \dfrac{x}{x^2+1}, \ x \in \mathbb{R}$.

12. Show that the equation $\tan x = 1 - x$ has at least one solution for $0 < x < 1$.

13. Show that there is no $k \in \mathbb{R}$ such that the equation $x^3 + 12x + k = 0$ has two distinct roots in $[0, 2]$.

14. Let I be an interval and $f : I \to \mathbb{R}$ be differentiable on I. Show that if the derivative f' is never 0 on I, then either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

15. Let p be a polynomial of degree $n \geq 2$. Show that between any two distinct roots of p there lies a root of p'.

L' Hospital's Rules:

In calculus, we frequently come across limits of the form $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$. We note that the limit exists and is equal to $\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim\limits_{x\to a} f(x), \ \lim\limits_{x\to a} g(x)$ both exist and $\lim\limits_{x\to a} g(x) \neq 0$.

But if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then as $x \to a$, $\dfrac{f(x)}{g(x)}$ may tend to a finite limit or to ∞ or to $-\infty$ or it may oscillate.

In this case, we say that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is of indeterminate form $\dfrac{0}{0}$.

Similarly, if $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is said to be of the indeterminate form $\dfrac{\infty}{\infty}$.

We state below a collection of theorems called L' Hospital's rule which are useful in evaluating limits of the form $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$. Other indeterminate forms like $0 \cdot \infty, \ 0^0, \ 1^\infty, \ \infty - \infty$ can be reduced either to $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ form by algebraic manipulations and by taking logarithms or exponentials.

Theorem 10 *: **(L' Hospital's Rule I)***
Suppose $a < b$ and functions f, g are derivable on (a, b) and $g'(x) \neq 0 \quad \forall x \in (a, b)$. Suppose that $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$ and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.
Here, L can be a real number or ∞ or $-\infty$.

Remarks:

1. L' Hospital's rule for $\frac{0}{0}$ indeterminate form is also valid for left land limits. There we replace (a, b) by (c, a) where $c < a$ and $x \to a^+$ by $x \to a^-$.
   Combining the versions of left hand limits and right hand limits, we obtain L' Hospital's rule for (two sided) limits of $\frac{0}{0}$ indeterminate form as follows:
   Let f, g be functions which are differentiable in some deleted neighborhood of a where $g'(x) \neq 0$ and let $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.
2. L' Hospital's rule I is also valid if instead of considering limits as $x \to a$ where a is a real number, we consider limits as $x \to \infty$ or $x \to -\infty$.
3. If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ is also indeterminate of the form $\frac{0}{0}$, then it may be possible to apply the above theorem again to conclude that if $\lim_{x \to a^+} \frac{f''(x)}{g''(x)}$ exists, then it is equal to $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$.

Theorem 11 *: **(L' Hospital's Rule II)***
Suppose $a < b$ and functions f, g are derivable on (a, b) and $g'(x) \neq 0 \quad \forall x \in (a, b)$. Suppose that $\lim_{x \to a^+} f(x) = \pm\infty = \lim_{x \to a^+} g(x)$ and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Remarks:

1. Above L' Hospital's rule can be stated for $x \to a^-$ or $x \to a$ with suitable modifications in the hypothesis of the theorem.

2. L' Hospital's rule II is also valid if instead of considering limits as $x \to a$ where a is a real number, we consider limits as $x \to \infty$ or $x \to -\infty$.

3. If $\lim\limits_{x \to a^+} \dfrac{f'(x)}{g'(x)}$ is again indeterminate of the form $\dfrac{\infty}{\infty}$, then by applying the above theorem again, we conclude that, if $\lim\limits_{x \to a^+} \dfrac{f''(x)}{g''(x)}$ exists, then it is equal to $\lim\limits_{x \to a^+} \dfrac{f'(x)}{g'(x)}$.

Examples:

1. Find $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.
   Solution: $f(x) = \sin x,\ g(x) = x$ are differentiable everywhere on the real line and $f'(x) = \cos x,\ g'(x) = 1$.
   Further, $g'(x) \neq 0$ for all $x \in \mathbb{R}$.
   $\lim\limits_{x \to 0} f(x) = 0 = \lim\limits_{x \to 0} g(x)$
   and $\lim\limits_{x \to 0} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} \cos x = 1$.
   Therefore, by L' Hospital's rule I, we have
   $$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 1.$$

2. Show that $\lim\limits_{x \to \infty} \dfrac{x^2}{e^{2x}} = 0$.

   Solution: Note that $\lim\limits_{x \to \infty} \dfrac{x^2}{e^{2x}}$ is of the indeterminate form $\dfrac{\infty}{\infty}$. Also, $f(x) = x^2, g(x) = e^{2x}$ satisfy all hypotheses of L' Hospital's rule II, so $\lim\limits_{x \to \infty} \dfrac{x^2}{e^{2x}}$ exists provided $\lim\limits_{x \to \infty} \dfrac{2x}{2e^{2x}}$ exists. and again by L' Hospital's rule, this limit exists provided $\lim\limits_{x \to \infty} \dfrac{2}{4e^{2x}}$ exists.
   Now $\lim\limits_{x \to \infty} \dfrac{2}{4e^{2x}} = 0$ and hence $\lim\limits_{x \to \infty} \dfrac{x^2}{e^{2x}} = 0$.

3. Evaluate $\lim\limits_{x\to\infty} \dfrac{2x^2+5x+7}{x^2-x+1}$.

Solution: $\lim\limits_{x\to\infty} \dfrac{2x^2+5x+7}{x^2-x+1}$ $\quad\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x\to\infty} \frac{4x+5}{2x-1} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{4}{2}$$

$$= 2.$$

4. Find $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$.

Solution: Let $f(x) = 1-\cos x$ and $g(x) = x^2$.
Both f, g are differentiable in every deleted neighborhood of 0 and $g'(x) = 2x \neq 0$ in that deleted neighborhood.
Also, $\lim\limits_{x\to 0} f(x) = 0 = \lim\limits_{x\to 0} g(x)$.
So, we can apply L' Hospital's rule I,

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos x}{2}$$

$$= 2.$$

Sometimes L' Hospital's rule does not help us to evaluate a given limit as illustrated in the following example.

5. Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{1+x^2}}{x}$.

Solution: Here, $f(x) = \sqrt{1+x^2}$ and $g(x) = x$.
Note that, $\lim\limits_{x\to\infty} f(x) = \infty = \lim\limits_{x\to\infty} g(x)$.
If we try to apply, L' Hospital's rule, we get a loop.

$$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} = \lim_{x\to\infty} \frac{\left(\frac{2x}{2\sqrt{1+x^2}}\right)}{1}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{1+x^2}}.$$

$$= \lim_{x\to\infty} \frac{1}{\left(\frac{2x}{2\sqrt{1+x^2}}\right)}.$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x}.$$

However, the limit can be found directly as,

$$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} = \lim_{x\to\infty} \sqrt{\frac{1+x^2}{x^2}} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} = 1.$$

6. Find $\lim\limits_{x\to\infty} \dfrac{\log x}{x}$.
Solution: Here $f(x) = \log x$ and $g(x) = x$ are differentiable on $(0, \infty)$ and $g'(x) = 1 \neq 0,\ \forall x \in (0, \infty)$.
By L' Hospital's rule, we have $\lim\limits_{x\to\infty} \dfrac{\log x}{x} = \lim\limits_{x\to\infty} \dfrac{\left(\frac{1}{x}\right)}{1} = 0$.

7. Find $\lim\limits_{x\to 0} \dfrac{e^x - \cos x - x}{x \sin x}$.
Solution: $\lim\limits_{x\to 0} \dfrac{e^x - \cos x - x}{x \sin x}$ $\quad\left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{e^x - \cos x - x}{x^2} \frac{x}{\sin x}$$
$$= \lim_{x\to 0} \frac{e^x - \cos x - x}{x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$
$$= \lim_{x\to 0} \frac{e^x + \sin x - 1}{2x} \qquad \left(\frac{0}{0} \text{ form}\right)$$
$$= \lim_{x\to 0} \frac{e^x + \cos x}{2}$$
$$= 1.$$

Now, we see how to evaluate limits for other indeterminate forms such as $\infty - \infty,\ 0 \cdot \infty,\ 1^\infty,\ 0^0,\ \infty^0$. Here, the first two forms can be converted into $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$. For the remaining forms we need to use the continuity of the logarithmic and exponential functions.

8. Evaluate $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$.
Solution: $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$ has indeterminate form $\infty - \infty$.

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \frac{e^x - 1 - x}{x(e^x - 1)} \qquad \left(\frac{0}{0} \text{ form}\right)$$
$$= \frac{e^x - 1}{e^x - 1 + xe^x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{e^x}{2e^x + xe^x}$$
$$= \frac{1}{2}.$$

9. Evaluate $\lim_{x\to\pi/2}(\sec x - \tan x)$.

Solution: $\lim_{x\to\pi/2}(\sec x - \tan x)$ $\quad (\infty - \infty \text{ form})$

$$= \lim_{x\to\pi/2}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$
$$= \lim_{x\to\pi/2}\left(\frac{1-\sin x}{\cos x}\right) \qquad \left(\frac{0}{0} \text{ form}\right)$$
$$= \lim_{x\to\pi/2}\frac{\cos x}{-\sin x}$$
$$= 0.$$

10. Evaluate $\lim_{x\to 0^+} x^2 \log x$.

Solution: $\lim_{x\to 0} x^2 \log x$ $\quad (0 \cdot \infty \text{ form})$

$$= \lim_{x\to 0}\frac{\log x}{1/x^2} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$
$$= \lim_{x\to 0}\frac{1/x}{-2/x^3}$$
$$= \lim_{x\to 0}\frac{-x^2}{2}$$
$$= 0.$$

11. Evaluate $\lim_{x\to 2}(2-x)\tan\left(\frac{\pi x}{4}\right)$.

Solution: $\lim_{x\to 2}(2-x)\tan\left(\frac{\pi x}{4}\right)$ $\quad (0 \cdot \infty \text{ form})$

$$= \lim_{x\to 2}\frac{2-x}{\cot\left(\frac{\pi x}{4}\right)} \qquad \left(\frac{0}{0} \text{ form}\right)$$
$$= \lim_{x\to 2}\frac{-1}{-\frac{\pi}{4}\operatorname{cosec}^2\left(\frac{\pi x}{4}\right)}$$
$$= \lim_{x\to 2}\frac{4}{\pi}\sin^2\left(\frac{\pi x}{4}\right)$$
$$= \frac{4}{\pi}.$$

12. Evaluate $\lim_{x\to 0^+} x^x$ $\quad (0^0 \text{ form})$

Solution: Put $y = x^x$.

$\therefore \log y = x \log x \ \to 0$ as $x \to 0^+$.

Hence, $\displaystyle\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\log y}$

$$= e^{\lim_{x\to 0^+} \log y} \quad \text{(using continuity of } e^t)$$

$$= e^0 = 1.$$

$\therefore \displaystyle\lim_{x\to 0^+} x^x = 1.$

13. Evaluate $\displaystyle\lim_{x\to 0^+} (1+mx)^{1/x}$ (1^∞ form)

Solution: Put $y = (1+mx)^{1/x}$.

$\therefore \log y = \dfrac{1}{x}\log(1+mx).$

$$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{1}{x}\log(1+mx) \quad (0\cdot\infty \text{ form})$$
$$= \lim_{x\to 0^+} \frac{\log(1+mx)}{x}$$
$$= \lim_{x\to 0^+} \frac{m}{1+mx}$$
$$= m.$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\log y}$$
$$= e^{\lim_{x\to 0^+} \log y} = e^m.$$

$\therefore \displaystyle\lim_{x\to 0^+} (1+mx)^{1/x} = e^m.$

14. Evaluate $\displaystyle\lim_{x\to 0^+} x^{\sin x}$ (0^0 form)

Solution: Let $y = x^{\sin x}$.

$\therefore \log y = \sin x \log x.$

$$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \sin x \log x \quad (0\cdot\infty \text{ form})$$
$$= \lim_{x\to 0^+} \frac{\log x}{\text{cosec } x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$
$$= \lim_{x\to 0^+} \frac{1/x}{\text{cosec } x \cot x}$$
$$= \lim_{x\to 0^+} \frac{\sin^2 x}{x\cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$
$$= \lim_{x\to 0^+} \frac{2\sin x\cos x}{\cos x - x\sin x}$$
$$= 0.$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\log y}$$
$$= e^{\lim\limits_{x\to 0^+} \log y}$$
$$= e^0 = 1.$$
$$\therefore \lim_{x\to 0^+} x^{\sin x} = 1.$$

15. Evaluate $\lim\limits_{x\to 0} (\operatorname{cosec} x)^{1/\log x}$ (∞^0 form)

Solution: Let $y = (\operatorname{cosec} x)^{1/\log x}$.

$$\therefore \log y = \frac{1}{\log x} \log(\operatorname{cosec} x).$$
$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log(\operatorname{cosec} x)}{\log x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$
$$= \lim_{x\to 0} \frac{\frac{-1}{\operatorname{cosec} x} \operatorname{cosec} x \cdot \cot x}{\frac{1}{x}}$$
$$= -\lim_{x\to 0} x \cot x$$
$$= -\lim_{x\to 0} \cos x \cdot \frac{x}{\sin x}$$
$$= -1 \cdot 1 = -1.$$
$$\therefore \lim_{x\to 0} y = \lim_{x\to 0} e^{\log y}$$
$$= e^{\lim\limits_{x\to 0} \log y}$$
$$= e^{-1} = \frac{1}{e}.$$
$$\therefore \lim_{x\to 0} (\operatorname{cosec} x)^{1/\log x} = \frac{1}{e}.$$

16. Evaluate $\lim\limits_{x\to 0} (a^x + x)^{1/x}$ (1^∞ form)

Solution: Let $y = (a^x + x)^{1/x}$.

$$\therefore \log y = \frac{1}{x} \log(a^x + x)$$
$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{1}{x} \log(a^x + x) \qquad (\infty \cdot 0 \text{ form})$$
$$= \lim_{x\to 0} \frac{\log(a^x + x)}{x} \qquad \left(\frac{0}{0} \text{ form}\right)$$
$$= \lim_{x\to 0} \frac{\frac{1}{a^x + x}(a^x \log a + 1)}{1}$$
$$= \lim_{x\to 0} \frac{a^x \log a + 1}{a^x + x}$$

$$
\begin{aligned}
&= \log a + 1. \\
\therefore \lim_{x\to 0} y &= \lim_{x\to 0} e^{\log y} \\
&= e^{\lim\limits_{x\to 0} \log y} \\
&= e^{\log a + 1} \\
&= e^{\log a} \cdot e^{1} = a \cdot e. \\
\therefore \lim_{x\to 0} (a^x + x)^{1/x} &= a \cdot e.
\end{aligned}
$$

Exercises:

1. Evaluate the following limits, if they exist.

   (a) $\displaystyle \lim_{x\to\infty} \frac{x^2}{e^{2x}}$

   (b) $\displaystyle \lim_{x\to 0} \frac{e^{2x} - \cos x}{x}$

   (c) $\displaystyle \lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$

   (d) $\displaystyle \lim_{x\to 0} \frac{1 + x}{x}$

   (e) $\displaystyle \lim_{x\to 0} \frac{x}{\tan x}$

   (f) $\displaystyle \lim_{x\to 0} \frac{2x^2 - 3x + 1}{x^2 - x}$

   (g) $\displaystyle \lim_{x\to 0} \frac{e^x - 1 + \log(1 - x)}{\tan x - x}$

   (h) $\displaystyle \lim_{x\to\infty} x^{\frac{1}{x}}$

   (i) $\displaystyle \lim_{x\to 0^+} \frac{\tan 5x}{\tan x}$

   (j) $\displaystyle \lim_{x\to 0} \frac{a^x - 1}{b^x - 1}$

   (k) $\displaystyle \lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$

   (l) $\displaystyle \lim_{x\to 1} (1 - x)\frac{\tan \pi x}{2}$

(m) $\lim_{x \to 0} \left(\frac{1}{x} - \cot x \right)$

(n) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{\log(1+x)}{x^2} \right)$

(o) $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

(p) $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

(q) $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$

(r) $\lim_{x \to \infty} \left(1 + \frac{3}{x} \right)^x$

(s) $\lim_{x \to 1} \left(x^2 - 1 \right)^{\log x}$

(t) $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$

(u) $\lim_{x \to a} (x-a)^{x-a}$

(v) $\lim_{x \to \pi/2} (\sec x)^{\cot x}$

(w) $\lim_{x \to 0^+} \left(\frac{1}{x} \right)^{2 \sin x}$

(x) $\lim_{x \to \infty} (e^x + x)^{\frac{1}{x}}$

2. Show that

(a) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

(b) $\lim_{x \to a} \frac{a^x - b^x}{x} = \log a - \log b$ for $a, b > 0$.

(c) $\lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} = \sqrt{ab}$.

(d) $\lim_{x \to \infty} (1 + mx)^{\frac{1}{x}} = e$ for $m > 0$.

3. Find the value of a, b such that

$$\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1.$$

Successive Differentiation:

Derivatives of order greater than one are obtained by a natural extension of the differentiation process, called successive differentiation. If the derivative $f'(x)$ of a function f exists at every point x in an interval I containing a point c, then we can consider existence of the derivative of the function f' at the point c. In case f' has a derivative at the point c, we refer to the resulting number as the second derivative of f at c, and we denote this number by $f''(c)$ or by $f^{(2)}(c)$. In similar fashion, we define the third derivative $f'''(x) = f^{(3)}(c), \ldots,$ and the n^{th} derivative $f^{(n)}(c)$, whenever these derivatives exist. Note that the existence of the n^{th} derivative presumes the existence of the $(n-1)^{th}$ derivative in an interval containing c.
If $y = f(x)$, we denote the n^{th} derivative of f at x as $f^{(n)}(x)$ or $y_n(x)$ or $D^n f(x)$ or $\dfrac{d^n y}{dx^n}$.

Examples:

1. If $y = \sin^{-1} x$, prove that $(1-x^2)y_2 - xy_1 = 0$.
   Solution: $y = \sin^{-1} x$.
   $\therefore y_1 = \dfrac{1}{\sqrt{1-x^2}}$
   $\therefore y_1^2 = \dfrac{1}{1-x^2}$
   $\therefore (1-x^2)y_1^2 - 1 = 0.$
   Differentiating both sides w.r.t. x, we get

   $$(1-x^2)2y_1y_2 + y_1^2(-2x) = 0$$

   $$\therefore (1-x^2)y_2 - xy_1 = 0$$

   .

2. If $y = \log(\log x)$, prove that $xy_2 + y_1 + xy_1^2 = 0$.
   Solution: $y = \log(\log x)$.

$\therefore y_1 = \dfrac{1}{\log x}\dfrac{1}{x}$

$\therefore xy_1 = \dfrac{1}{\log x}$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned} xy_2 + y_1 &= -\frac{1}{(\log x)^2}\frac{1}{x} \\ &= -\frac{x}{x^2(\log x)^2} \\ &= -xy_1^2. \end{aligned}$$

$\therefore xy_2 + y_1 + xy_1^2 = 0.$

The n^{th} derivative of some standard functions:

1. If $y = (ax+b)^m,\ m \in \mathbb{R}$,
   then $y_1 = ma(ax+b)^{m-1}$,
   $y_2 = m(m-1)a^2(ax+b)^{m-2}$,
   $y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$ and
   in general, $y_n = m(m-1)(m-2)\dots(m-n+1)a^n(ax+b)^{m-n}$.
   If m is a positive integer, then
   $$\begin{aligned} y_n &= \frac{m!}{(m-n)!}a^n(ax+b)^{m-n} && \text{if } n < m \\ &= m!a^m && \text{if } n = m \\ &= 0 && \text{if } n > m. \end{aligned}$$

2. If $y = \dfrac{1}{ax+b}$.
   Put $m = -1$ in (1), we get
   $$\begin{aligned} y_n &= (-1)(-2)\dots(-n)a^n(ax+b)^{-1-n} \\ &= \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}. \end{aligned}$$

3. If $y = \log(ax+b)$.
   We have, $y_1 = \dfrac{1}{ax+b}\cdot a$.
   By above result (2), we have
   $$y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}.$$

4. If $y = a^{mx}$.
   We have $y_1 = ma^{mx}\log a$

$y_2 = m^2 a^{mx} (\log a)^2$
$y_3 = m^3 a^{mx} (\log a)^3$.
In general,
$y_n = m^n a^{mx} (\log a)^n$.

5. If $y = e^{mx}$.
   Put $a = e$ in (4), we get
   $y_n = m^n e^{mx} (\log e)^n$
   $= m^n e^{mx}$.

6. If $y = \sin(ax + b)$, then we have
   $y_1 = a\cos(ax + b) = a\sin(ax + b + \pi/2)$
   $y_2 = a^2\cos(ax + b + \pi/2) = a^2\sin(ax + b + 2\pi/2)$
   $y_3 = a^3\cos(ax + b + 2\pi/2) = a^3\sin(ax + b + 3\pi/2)$
   In general,
   $y_n = a^n \sin(ax + b + n\pi/2)$.

7. Similar to (6), if $y = \cos(ax + b)$, then
   $y_n = a^n \cos(ax + b + n\pi/2)$.

8. If $y = e^{ax}\sin(bx + c)$, we have
   $y_1 = ae^{ax}\sin(bx + c) + be^{ax}\cos(bx + c)$.
   Put $a = r\cos\theta,\ b = r\sin\theta$
   where $r = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}\dfrac{b}{a}$.
   $\therefore y_1 = re^{ax}\cos\theta\sin(bx + c) + re^{ax}\sin\theta\cos(bx + c)$
   $= re^{ax}\sin(bx + c + \theta)$
   $\therefore y_2 = r^2 e^{ax}\sin(bx + c + 2\theta)$
   In general,
   $y_n = r^n e^{ax}\sin(bx + c + n\theta)$.

9. Similarly, if $y = e^{ax}\cos(bx + c)$, then
   $y_n = r^n e^{ax}\cos(bx + c + n\theta)$
   where $r = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}\dfrac{b}{a}$.

Examples:

1. If $y = \dfrac{16x+18}{2x^2+5x+3}$, find y_n.
Solution: Here $y = \dfrac{16x+18}{2x^2+5x+3} = \dfrac{16x+18}{(x+1)(2x+3)}$
$\therefore y = \dfrac{2}{x+1} + \dfrac{12}{2x+3}$
$\therefore y_n = \dfrac{2(-1)^n n!1^n}{(x+1)^{n+1}} + \dfrac{12(-1)^n n!2^n}{(2x+3)^{n+1}}$.

2. If $y = e^{2x}\cos^2 x \sin x$, find the n^{th} derivative of y.
Solution: We have
$$
\begin{aligned}
y &= e^{2x}\cos^2 x\sin x \\
&= e^{2x}\frac{1}{2}(1+\cos 2x)\sin x \\
&= e^{2x}\left(\frac{1}{2}\sin x + \frac{1}{2}\cos 2x\sin x\right) \\
&= e^{2x}\left(\frac{1}{2}\sin x + \frac{1}{4}\sin 3x - \frac{1}{4}\sin x\right) \\
&= e^{2x}\left(\frac{1}{4}\sin x + \frac{1}{4}\sin 3x\right)
\end{aligned}
$$
$\therefore y = \dfrac{1}{4}e^{2x}\sin x + \dfrac{1}{4}e^{2x}\sin 3x$.
$\therefore y_n = \dfrac{1}{4}\sqrt{5}^n\, e^{2x}\sin\left(x + n\tan^{-1}\dfrac{1}{2}\right) + \dfrac{1}{4}\sqrt{13}^n\, e^{2x}\sin\left(3x + n\tan^{-1}\dfrac{3}{2}\right)$.

Theorem 12 *: **(Leibnitz's Theorem)***
If $y = uv$, where u and v are functions of x possessing derivatives of n^{th} order, then

$$
\begin{aligned}
y_n &= (uv)_n \\
&= {}^nC_0u_nv + {}^nC_1u_{n-1}v_1 + \ldots + {}^nC_{r-1}u_{n-r+1}v_{r-1} \\
&\qquad + {}^nC_ru_{n-r}v_r + \ldots + {}^nC_nuv_n \qquad \text{[I]}
\end{aligned}
$$

where ${}^nC_r = \dfrac{n!}{(n-r)!r!}$.

Proof: We prove the theorem using mathematical induction.
Step 1: $y_1 = (uv)_1$
$= u_1v + uv_1$

$$= {}^1C_0u_1v + {}^1C_1uv_1$$

Thus, the result is true for $n = 1$.

Step 2: Assume that the result is true for $n = m$. Therefore

$$\begin{aligned}
y_m &= (uv)_m \\
&= {}^m C_0u_mv + {}^m C_1u_{m-1}v_1 + \ldots + {}^m C_{r-1}u_{m-r+1}v_{r-1} \\
&\qquad + {}^m C_ru_{m-r}v_r + \ldots + {}^m C_{m-1}u_1v_{m-1} + {}^m C_muv_m.
\end{aligned}$$

Differentiating both sides, we get

$$\begin{aligned}
y_{(m+1)} &= (uv)_{m+1} \\
&= {}^m C_0u_{m+1}v + {}^m C_0u_mv_1 + {}^m C_1u_mv_1 + {}^m C_1u_{m-1}v_2 \\
&\qquad + \ldots + {}^m C_{r-1}u_{m-r+2}v_{r-1} + {}^m C_{r-1}u_{m-r+1}v_r \\
&\qquad + {}^m C_ru_{m-r+1}v_r + {}^m C_ru_{m-r}v_{r+1} \\
&\qquad + \ldots + {}^m C_mu_1v_m + {}^m C_muv_{m+1} \\
&= {}^m C_0u_{m+1}v + ({}^mC_0 + {}^m C_1)u_mv_1 + ({}^mC_1 + {}^m C_1)u_{m-1}v_2 \\
&\qquad + \ldots + ({}^mC_{r-1} + {}^m C_r)u_{m-r}v_r + \ldots + {}^m C_muv_{m+1}.
\end{aligned}$$

We have the results,

$${}^mC_{r-1} + {}^m C_r = {}^{m+1} C_r \text{ and } {}^mC_m = {}^{m+1} C_{m+1},\ {}^mC_0 = {}^{m+1} C_0.$$

Therefore

$$\begin{aligned}
y_{m+1} &= (uv)_{m+1} \\
&= {}^{m+1} C_0u_{m+1}v + {}^{m+1} C_1u_mv_1 + {}^{m+1} C_2u_{m-1}v_2 \\
&\qquad + \ldots + {}^{m+1} C_ru_{m-r+1}v_r + \ldots + {}^{m+1} C_{m+1}uv_{m+1}.
\end{aligned}$$

Therefore, the result is true for $n = m + 1$.
Hence, by the principle of mathematical induction the theorem is true for all positive integers n.

Examples:

1. If $y = (\sin^{-1} x)^2$, prove that
   $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$.

Solution: We have $y = (\sin^{-1} x)^2$.
Differentiating both sides w.r.t. x, we get
$y_1 = 2\sin^{-1} x \cdot \dfrac{1}{\sqrt{1-x^2}}$
$\therefore y_1\sqrt{1-x^2} = 2\sin^{-1} x$.
Squaring and simplifying, we get
$(1-x^2)y_1^2 - 4y = 0$.
Differentiating once again, we get
$(1-x^2)2y_1y_2 - 2xy_1^2 - 4y_1 = 0$
$\therefore 2y_1\left[(1-x^2)y_2 - xy_1 - 2\right] = 0$
$\therefore (1-x^2)y_2 - xy_1 - 2 = 0$ [I]
Differentiating [I] n times using Leibnitz's theorem, we get

$$\left[{}^nC_0(1-x^2)y_{n+2} +^n C_1(-2x)y_{n+1} +^n C_2(-2)y_n\right] - [{}^nC_0xy_{n+1} +^n C_1y_n] = 0$$

$\therefore (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = 0$
$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$.

2. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, show that
$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.
Solution: We have $y^{\frac{1}{m}} + \dfrac{1}{y^{\frac{1}{m}}} = 2x$.
Put $y^{\frac{1}{m}} = t$.
$\therefore t + \dfrac{1}{t} = 2x$.
$\therefore t^2 - 2xt + 1 = 0$ which is a quadratic equation in t. Solving it for t, we get
$t = x \pm \sqrt{x^2-1}$.
We ignore the negative sign.
$\therefore t = x + \sqrt{x^2-1} = y^{\frac{1}{m}}$
$\therefore y = (x+\sqrt{x^2-1})^m$ [I]
Differentiating [I] w.r.t. to x, we get
$y_1 = m(x+\sqrt{x^2-1})^{m-1}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$
$\therefore y_1 = \dfrac{m(x+\sqrt{x^2-1})^m}{\sqrt{x^2-1}}$
$\therefore \sqrt{x^2-1}y_1 = my$

Differentiating once again, we get
$\sqrt{x^2-1}y_2 + \dfrac{x}{\sqrt{x^2-1}}y_1 = my_1$
$= m\frac{my}{\sqrt{x^2-1}}$
$\therefore (x^2-1)y_2 + xy_1 - m^2y = 0$ $[II]$
Differentiating $[II]$ n times using Leibnitz's theorem, we get
$(x^2-1)y_{n+2} + n(2x)y_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2y_n = 0$
$\therefore (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$

3. If $x = \tan(\log y)$, prove that
$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0.$
Solution: We have $x = \tan(\log y)$
$\therefore \log y = \tan^{-1} x$
$\therefore y = e^{\tan^{-1} x}$ $[I]$
Differentiating $[I]$ w.r.t. x, we get
$y_1 = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$
$\therefore (1+x^2)y_1 = e^{\tan^{-1} x} = y$
$\therefore (1+x^2)y_1 - y = 0$ $[II]$
Differentiating $[II]$ n times using Leibnitz's theorem, we get
$(1+x^2)y_{n+1} + n(2x)y_n + n(n-1)y_{n-1} - y_n = 0$
$\therefore (1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$

Exercises:

1. Find the n^{th} derivative of
   (a) $x^n e^x$
   (b) $x^3 \cos x$

2. Let $f(x) = x \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that f is continuous at 0 but $f'(0)$ does not exist.

3. Let $f(x) = x^2 \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(0)$ exists but f' is not continuous at 0.

4. Prove that

(a) $\dfrac{d^n}{dx^n}\left(\dfrac{1}{a^2-x^2}\right) = \dfrac{n!}{2a}\left[\dfrac{(-1)^{n+1}}{(a-x)^{n+1}} + \dfrac{(-1)^n}{(a+x)^{n+1}}\right].$

(b) If $y = \sin^2 x \sin 2x$, then
$y_n = \dfrac{1}{4}\left[2^{n+1}\sin(2x + n\pi/2) - 4^{n-1}\sin(4x + n\pi/2)\right].$

(c) If $y = e^{ax}\cos x \sin x$ then
$y_n = \dfrac{1}{2}e^{ax}(a^2+4)^{n/2}\sin(2x + n\tan^{-1}\left(\dfrac{2}{a}\right)).$

5. If $y = e^{a\sin^{-1}x}\cos x \sin x$ then prove that
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$

6. If $y = \sin(m\sin^{-1}x)$, then show that
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$

7. If $y = (x^2-1)^n$, prove that
$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$

8. If $y = \log(x + \sqrt{1+x^2})$, prove that

   (a) $(1+x^2)y_2 + xy_1 = 0$

   (b) $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

   (c) $y_{n+2}(0) = -n^2y_n(0)$.

Taylor's Theorem:
Taylor's theorem can be regarded as an extension of the mean value theorem to higher order derivatives. Whereas the mean value theorem relates the difference of values of a function and its first derivative, Taylor's theorem provides a relation between the difference of values of the function and its higher order derivatives.
It is useful to be able to approximate a function by an infinite series. Taylor's theorem can be used to represent a function as an infinite series.

Theorem 13 *: **(Taylor's Theorem with Lagrange's form of remainder)***
Let $n \in \mathbb{N}$, let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be such that f and its derivatives

$f', f'', \ldots f^{(n)}$ are continuous on I and that $f^{(n+1)}$ exists on (a,b). If $x_0 \in I$, then for any $x \in I$, there exists a point c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Note: We may write the conclusion of Taylor's theorem as
$f(x) = P_n(x) + R_n(x)$ where
$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ is the Taylor polynomial for f at x_0 and $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ for some point c between x and x_0, is the Lagrange's form of remainder of order n.
Suppose f possesses derivatives of all orders on I. If $R_n(x) \to 0$ as $n \to \infty$ for all x in I, then we say that the Taylor series generated by f at x_0 converges to f on I and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

This series is known as Taylor's series expansion of f at x_0.

With $x_0 = a$ and $x = a + h$, we have the following version of Taylor's theorem.

Theorem 14 *: **(Taylor's Theorem - Second form)***
Let f be a function defined on $[a, a+h]$ such that

1. *the n^{th} derivative $f^{(n)}$ is continuous on $[a, a+h]$*
2. *the $(n+1)^{th}$ derivative $f^{(n+1)}$ exists on $(a, a+h)$.*

Then there exists a point c in $(a, a+h)$ such that

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ldots + \frac{f^{(n)}(a)}{n!}h^n + \frac{f^{(n+1)}(c)}{(n+1)!}h^{n+1}.$$

We have another form of Taylor's theorem known as Maclaurin's theorem.

Theorem 15 *: **(Maclaurin's Theorem)***
Let f be a function defined on $[0, x]$ such that

1. *the n^{th} derivative $f^{(n)}$ is continuous on $[0, x]$*
2. *the $(n+1)^{th}$ derivative $f^{(n+1)}$ exists on $(0, x)$.*

Then there exists at least one real number c in $(0, x)$ such that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ is called the Lagrange's form of remainder.

Note: Suppose f possesses derivatives of all orders on I. If $R_n(x) \to 0$ as $n \to \infty$ for all x in I, then the Maclaurin series generated by f converges to f and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.$$

This series is known as Maclaurin's series expansion of f.

We are not proving any of these theorems. Assuming their validity we solve examples.

Maclaurin's series expansions of some basic functions:

1. Expansion of e^x
   Let $f(x) = e^x \quad \therefore f(0) = e^0 = 1$
   $f'(x) = e^x \quad \therefore f'(0) = 1$
   $f''(x) = e^x \quad \therefore f''(0) = 1$
   $\vdots$
   $f^{(n)}(x) = e^x \quad \therefore f^{(n)}(0) = 1$
   By Maclaurin's theorem, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$
$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

2. Expansion of $\sin x$
   Let $f(x) = \sin x \quad \therefore f(0) = 0$
   $f'(x) = \cos x \quad \therefore f'(0) = 1$
   $f''(x) = -\sin x \quad \therefore f''(0) = 0$
   $f^{(3)}(x) = -\cos x \quad \therefore f^{(3)}(0) = -1$
   $f^{(4)}(x) = \sin x \quad \therefore f^{(4)}(0) = 0$
   $\vdots$
   By Maclaurin's theorem, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$
$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

3. Expansion of $\cos x$
   Let $f(x) = \cos x \quad \therefore f(0) = 1$
   $f'(x) = -\sin x \quad \therefore f'(0) = 0$
   $f''(x) = -\cos x \quad \therefore f''(0) = -1$
   $f^{(3)}(x) = \sin x \quad \therefore f^{(3)}(0) = 0$
   $f^{(4)}(x) = \cos x \quad \therefore f^{(4)}(0) = 1$
   $\vdots$
   By Maclaurin's theorem, we have
$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$
$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

4. Expansion of $\log(1+x)$
   Let $f(x) = \log(1+x) \quad \therefore f(0) = \log 1 = 0$
   $f'(x) = \dfrac{1}{1+x} \quad \therefore f'(0) = 1$
   $f''(x) = \dfrac{-1}{(1+x)^2} \quad \therefore f''(0) = -1$

$f^{(3)}(x) = \dfrac{2}{(1+x)^3} \quad \therefore f^{(3)}(0) = 2$

$f^{(4)}(x) = \dfrac{-6}{(1+x)^4} \quad \therefore f^{(4)}(0) = -6$

$\vdots$

By Maclaurin's theorem, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

$$f(x) = \log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Examples:

1. Obtain Maclaurin's series expansion of $\tan x$.

   Solution: Let $y = f(x) = \tan x \quad \therefore y(0) = 0$

   $f'(x) = \sec^2 x = 1 + \tan^2 x$

   $\therefore y_1 = 1 + y^2 \quad \therefore y_1(0) = 1$

   $y_2 = 2yy_1 \quad \therefore y_2(0) = 0$

   $y_3 = 2y_1^2 + 2yy_2 \quad \therefore y_3(0) = 2$

   $y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 \quad \therefore y_4(0) = 0$

   $y_5 = 8y_1y_3 + 6y_2^2 + 2yy_4 \quad \therefore y_5(0) = 16$

   $\vdots$

   By Maclaurin's theorem, we have

   $$f(x) = y(0) + y_1(0)x + \frac{y_2(0)}{2!}x^2 + \ldots + \frac{y_{(n)}(0)}{n!}x^n + \ldots$$

   $$f(x) = \tan x = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \ldots$$

   $$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$$

2. Verify the Taylor's series expansion

   $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Hence find the value of $\frac{\pi}{4}$ as an infinite series.

Solution: Here $y = f(x) = \tan^{-1} x$ $\quad\therefore y(0) = 0.$

$$y_1 = f'(x) = \frac{1}{1+x^2} \qquad \therefore y_1(0) = 1.$$

Now $(1+x^2)y_1 = 1$.

Differentiating n times using Leibnitz theorem this gives

$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0.$$

Hence at $x = 0$,

$$y_{n+1}(0) = -n(n-1)y_{n-1}(0).$$

Thus,

$$y_{2m}(0) = -(2m-1)(2m-2)y_{2m-2}(0) \qquad (1)$$

and

$$y_{2m+1}(0) = -(2m)(2m-1)y_{2m-1}(0) \qquad (2).$$

Since $y_0(0) = y(0) = 0$, by induction (1) shows that $y_{2m}(0) = 0$ for all m.

Since $y_1(0) = 1$, (2) gives successively $y_3(0) = -2, y_5(0) = 24, y_7(0) = -720$ and so on.

By Maclaurin's series, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

$$f(x) = \tan^{-1} x = x - \frac{2x^3}{3!} + \frac{24x^5}{5!} - \frac{720x^7}{7!} + \ldots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Putting $x = 1$, we have

$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

3. Assuming the validity of expansion prove that

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$$

Solution: Let $f(x) = e^x \cos x \quad \therefore f(0) = 1$
$f^{(n)}(x) = 2^{n/2} e^x \cos\left(x + \frac{n\pi}{4}\right)$
$\therefore f'(0) = \sqrt{2}\cos\left(\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$
$f''(0) = 2\cos\left(\frac{\pi}{2}\right) = 0$
$f^{(3)}(0) = 2\sqrt{2}\cos\left(\frac{3\pi}{4}\right) = 2\sqrt{2}\left(\frac{-1}{\sqrt{2}}\right) = -2$
$f^{(4)}(0) = 4\cos\pi = -4$
By Maclaurin's series expansion, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

$$f(x) = e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \ldots$$

4. Find series expansion of $\log\sqrt{\dfrac{1+x}{1-x}}$.
Solution: We have $\log\sqrt{\dfrac{1+x}{1-x}} = \dfrac{1}{2}[\log(1+x) - \log(1-x)]$.
$\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$ and
$\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$
Thus, $\log\sqrt{\dfrac{1+x}{1-x}} = \dfrac{1}{2}[\log(1+x) - \log(1-x)]$
$= \dfrac{1}{2}\left[\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\right) - \left(-x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\right)\right]$
$= \dfrac{1}{2}\left[2x + \dfrac{2x^3}{3} + \dfrac{2x^5}{5} + \dfrac{2x^7}{7} + \ldots\right]$
$= x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \ldots$

5. Use Taylor's theorem to express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$.
Solution: Let $f(x) = 2x^3 + 7x^2 + x - 6 \quad \therefore f(2) = 40$
$f'(x) = 6x^2 + 14x + 1 \quad \therefore f'(2) = 53$
$f''(x) = 12x + 14 \quad \therefore f''(2) = 38$
$f^{(3)}(x) = 12 \quad \therefore f^{(3)}(2) = 12.$

Hence $f^{(3)}$ is a constant function. So $f^{(4)}(x) \equiv 0$, for $n \geq 4$.
By Taylor's theorem,

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 + \dots$$
$$= 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$$

6. Expand $\sin x$ in ascending powers of $\left(x - \frac{\pi}{2}\right)$.
   Solution: Let $f(x) = \sin x$ $\quad \therefore f(\pi/2) = \sin(\pi/2) = 1$
   $f'(x) = \cos x$ $\quad \therefore f'(\pi/2) = 0$
   $f''(x) = -\sin x$ $\quad \therefore f''(\pi/2) = -1$
   $f^{(3)}(x) = -\cos x$ $\quad \therefore f^{(3)}(\pi/2) = 0$
   $f^{(4)}(x) = \sin x$ $\quad \therefore f^{(4)}(\pi/2) = 1$
   $\vdots$
   Using Taylor's theorem, we have

$$f(x) = f(\pi/2) + f'(\pi/2)(x-\pi/2) + \frac{f''(\pi/2)}{2!}(x-\pi/2)^2 + \frac{f^{(3)}(\pi/2)}{3!}(x-\pi/2)^3 + \dots$$
$$= 1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 + \dots$$

Exercises:

1. Expand $\tan x$ in powers of $\left(x - \frac{\pi}{4}\right)$.
2. Expand $5 + x^2 - 4x^4 + 3x^7$ in powers of $(x-1)$.
3. Find series expansion of $\sin^{-1} x$.
4. Use Taylor's theorem to expand the function $\frac{\log(1+x)}{1+x}$ in ascending powers of x upto first 4 terms.
5. Use Maclaurin's series to expand the function $\log(1 + \sin x)$.
6. Verify the Maclaurin's series expansion

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

7. Prove the Maclaurin's series expansion

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

8. Expand e^x in powers of $(x-1)$.

9. Find the Taylor's series generated by $f(x) = 2^x$ at $x_0 = 1$.

10. Find series expansion of $(1+x)^n$ on $[0, x]$.