Chapter 9
Cones and Cylinders

9.1 Cone

Definition 9.1 A surface generated by a straight line passing through a fixed point and intersecting a given curve is called as a cone. The fixed point is called the vertex of the cone and the given curve is called the guiding curve. A line which generates the cone is called a generator.

![Figure 9.1: Cone with a guiding curve](image)

The surface in the Fig. 9.1 is a cone with vertex $V$, The line $VA$ as a generator. The lines $VB, VC$ are also generators. In fact every line joining $V$ to any point of the guiding curve is a generator of the cone.

Remark 9.1 If the guiding curve is a plane curve of degree $n$, then the equation of the cone is also of degree $n$ and we call it a cone of order $n$.

In this chapter, we study only quadratic cones; i.e. cones having its equation of second degree in $x, y$ and $z$.

9.2 Equation of a cone

To find the equation of a cone with vertex $V(\alpha, \beta, \gamma)$ and whose guiding curve is the conic

$$\begin{align*} ax^2 + 2hxy + by^2 + 2gx + 2fy + c &= 0, \quad z = 0. \tag{9.1} \end{align*}$$

We have to find the locus of points on lines which pass through the vertex $V(\alpha, \beta, \gamma)$ and intersects the given guiding curve. Observe that the equations of any line passing through the vertex $V(\alpha, \beta, \gamma)$ and having direction ratios $l, m, n$ is given by,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \tag{9.2}$$

Any point on the equation (9.2) has the coordinates, $(\alpha + lt, \beta + mt, \gamma + nt)$, where $t \in \mathbb{R}$. If the line (9.2) intersects the guiding curve given by (9.1), then we have,

$$a(\alpha + lt)^2 + 2h(\alpha + lt)(\beta + mt) + b(\beta + mt)^2 + 2g(\alpha + lt) + 2f(\beta + mt) + c = 0, (\gamma + nt) = 0 \tag{9.3}$$

Eliminating $t, l, m$ and $n$ between (9.2) and (9.3), we get

$$a(\alpha z + \gamma x)^2 + 2h(\alpha z + \gamma x)(\beta z + \gamma y) + b(\beta z + \gamma y)^2 + 2g(\alpha z + \gamma x)(z - y) + 2f(\beta z + \gamma y)(z - y) + c(z - y)^2 = 0. \tag{9.4}$$

This is the required equation of the cone.

Remark 9.2 From the equation (9.4) it can be seen that the equation of a quadratic cone is of second degree in $x, y$ and $z$; i.e.,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \tag{9.5}$$
Example 9.1 Find the equation of a cone with vertex at the point (3,1,2) and guiding curve is \(2x^2 + 3y^2 = 1, z = 0\).

Solution The vertex is \(V(3,1,2)\). Let \(a, b, c\) be the direction ratios of a generator of the cone. Then the equations of generator are,

\[
\frac{x - 3}{a} = \frac{y - 1}{b} = \frac{z - 2}{c} = t \text{(say)} \tag{9.6}
\]

The coordinates of any point on the generator are \((3 + at, 1 + bt, 2 + ct)\).

For some \(t \in \mathbb{R}, (3 + at, 1 + bt, 2 + ct)\) lies on the guiding curve. Therefore \(2(3 + at)^2 + 3(1 + bt)^2 = 1\) and \(2 + ct = 0\). Thus, \(t = \frac{-2}{c}\). From this we get,

\[
2 \left(3 - 2a \right)^2 + 3 \left(1 - 2b \right)^2 = 1 \tag{9.6}
\]

From (9.6), we obtain \(2[3 - 2(\frac{2}{c})]^2 + 3[1 - 2(\frac{1}{c})]^2 = 1\) so

\[
2[3(z - 2) - 2(x - 3)]^2 + 3[1(z - 2) - 2(y - 1)]^2 = (z - 2)^2.
\]

After simplification, the required equation of the cone is,

\[
2x^2 + 3y^2 + 5z^2 - 3yz - 6xz + z - 1 = 0.
\]

9.3 Cone with vertex at the origin

Recall that the general equation of second degree equation is,

\[
ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \quad (9.7)
\]

We will show that \(u = v = w = d = 0\). Let \(P(x', y', z')\) be any point on the cone given by (9.7). Since \(O(0,0,0)\) is the vertex, the line \(OP\) is a generator of the cone. The equation of the line \(OP\) is given by,

\[
\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = t \text{(say)}
\]

Thus the coordinates of any point on the line \(OP\) are \(\left(\frac{xt}{t}, \frac{yt}{t}, \frac{zt}{t}\right)\).

The coordinates of any point on the generator are \(\left(\frac{3 + at}{t}, \frac{1 + bt}{t}, \frac{2 + ct}{t}\right)\) say \((9.6)\).

So that the equation of a cone with vertex at the origin is a homogeneous second degree equation.

**Theorem 9.1** The equation of a cone with vertex at the origin is a homogeneous second degree equation.

**Proof.** Let the equation of a quadratic cone with vertex at the origin be,

\[
ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \tag{9.7}
\]

We now claim that each of the constants \(u, v, w\) and \(d\) are zero. For if not i.e. if at least one of the constants is not zero, then the equation \(ux + vy + wz = 0\) would represent a plane and the point \((x', y', z')\) lies on the plane. As the point \((x', y', z')\) lies on the surface (9.7), it would mean that the surface (9.7) is a plane. This is impossible as it represents a cone. Hence \(u = 0, v = 0, w = 0\) and we also have \(d = 0\). So that the equation of a cone with vertex at the origin is of the form,

\[
ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,
\]

which is a homogeneous in \(x, y\) and \(z\) \(\blacksquare\).
9.3. Cone with vertex at the origin

Note that the converse of this theorem is also true.

**Theorem 9.2** Every second degree homogeneous equation in \(x, y, z\) represents a cone with vertex at the origin.

**Proof.** Consider the homogeneous equation of second degree in \(x, y, z\)

\[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad (9.9)\]

The coordinates of any point on the line are \((tx', ty', tz')\). Thus substituting these coordinates in \(L.H.S\) of (9.9) we have,

\[a(tx')^2 + b(ty')^2 + c(tz')^2 + 2f(ty')(tz') + 2g(tz')(tx') + 2h(tx')(ty') = t^2(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = t^2(0) \quad \ldots (9.9)\]

Hence, the point with coordinates \((tx', ty', tz')\) satisfies (9.9). Therefore any point on the line \(OP\) lies on the surface given by (9.9). Thus (9.9) is the surface generated by \(OP\). Therefore the equation (9.9) represents a cone with vertex at the origin.

**Remark 9.3** If the line with equation, \(\frac{x}{l} = \frac{y}{m} = \frac{z}{n}\) is a generator of a cone with vertex at the origin, then the direction ratios \(l, m, n\) satisfies the equation of the cone.

Let the equation of the cone be,

\[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.\]

The coordinates of any point on the line are \((lt, mt, nt)\), where \(t \in \mathbb{R}\). These coordinates satisfy the equation.

\[\therefore a(lt)^2 + b(mt)^2 + c(nt)^2 + 2f(mt)(nt) + 2g(nt)(lt) + 2h(lt)(mt) = 0\]

\[\therefore t^2(al^2 + bm^2 + cn^2 + 2fmn + 2gln + 2hlm) = 0.\]

This equation holds for all values of \(t\).

\[\therefore a(l)^2 + b(m)^2 + c(n)^2 + 2fmn + 2gln + 2hlm = 0.\]

Thus, it follows that the direction ratios of a generator satisfy the equation of the cone.

Also, it is easy to see that if the direction ratios \(l, m, n\) of a generator of a cone with vertex at the origin satisfy the equation \(\phi(l, m, n) = 0\), then the equation of the cone is \(\phi(x, y, z) = 0\). 

**Remark 9.4** The general equation of a quadratic cone with vertex at the origin is,

\[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.\]

Let the origin be shifted to the point \(V(\alpha, \beta, \gamma)\). In the new coordinate system the above equation becomes,

\[a(x - \alpha)^2 + b(y - \beta)^2 + c(z - \gamma)^2 + 2f(y - \beta)(z - \gamma) + 2g(z - \gamma)(x - \alpha) + 2h(x - \alpha)(y - \beta) = 0.\]

This is a general equation of a quadratic cone with vertex at \((\alpha, \beta, \gamma)\).
9.4 The Right circular Cone

Definition 9.2 A right circular cone is a surface generated by a straight line passing through a fixed point and making a constant angle $\theta$ with a fixed straight line passing through the given point. The fixed point is called as the vertex of the right circular cone, the fixed straight line is called as the axis of the cone and the constant angle $\theta$ is called as the semi-vertical angle of the cone.

Remark 9.5 Every section of a right circular cone by a plane perpendicular to its axis is a circle.

Let $\theta$ be the semi-vertical angle of the cone and $\alpha$ be a plane perpendicular to the axis $VN$ see Fig. 9.2 of the cone. We show that the section of the cone by the plane $\alpha$ is a circle.

Let $P$ be any point of the section of the cone by the plane $\alpha$. Let $A$ be the point of intersection of the axis $VN$ and the plane $\alpha$. Then $AP$ is perpendicular to $VA$. In the right angled triangle $VAP$, $\tan \theta = \frac{AP}{AV}$. $\therefore AP = AV \tan \theta$. Since, $AV$ and $\tan \theta$ are constant, it follows that $AP$ is constant for all points $P$ on the section of the cone by the plane $\alpha$. Thus the section is a circle.

9.4.1 Equation of a right circular cone

To find the equation of a right circular cone with a vertex $V(\alpha, \beta, \gamma)$ and whose axis is the line, $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$, with a semi-vertical angle $\theta$. Let $VN$ be the axis of the cone and let $P(x, y, z)$ be any point on the cone. The direction ratios of the generator $VP$ are $x - \alpha, y - \beta, z - \gamma$.

Let $P$ be any point of the section of the cone by the plane $\alpha$. Let $A$ be the point of intersection of the axis $VN$ and the plane $\alpha$. Then $AP$ is perpendicular to $VA$. In the right angled triangle $VAP$, $\tan \theta = \frac{AP}{AV}$. $\therefore AP = AV \tan \theta$. Since, $AV$ and $\tan \theta$ are constant, it follows that $AP$ is constant for all points $P$ on the section of the cone by the plane $\alpha$. Thus the section is a circle.

The angle between $VP$ and $VN$ is $\theta$ (see Fig. 9.3). As the direction ratios of the axis $VN$ are $l, m, n$, we have,

$$\cos \theta = \frac{l(x - \alpha) + m(y - \beta) + n(z - \gamma)}{\sqrt{l^2 + m^2 + n^2} \sqrt{(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2}}$$

Therefore, the equation of the right circular cone is,

$$[l(x - \alpha) + m(y - \beta) + n(z - \gamma)]^2 = (l^2 + m^2 + n^2) [(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2] \cos^2 \theta$$

The following example will make the above proof more clear.
Example 9.2 Find the equation of the right circular cone with vertex at 
\((2, -1, 4)\), the line \(\frac{x - 2}{1} = \frac{y + 1}{2} = \frac{z - 4}{-1}\) as the axis and semi-vertical angle \(\cos^{-1}(4/\sqrt{6})\).

**Solution.** Let \(P(x, y, z)\) be any point on the cone with vertex \(V(2, -1, 4)\). Then the direction ratios of a generator \(VP\) are \(x - 2, y + 1, z - 4\). The direction ratios of the axis \(\frac{x - 2}{1} = \frac{y + 1}{2} = \frac{z - 4}{-1}\) are \(1, 2, -1\). Let \(\theta\) be the semi-vertical angle. Hence, \(\cos \theta = \frac{4}{\sqrt{6}}\). From Remark, we have,

\[
\cos \theta = \frac{1(x - 2) + 2(y + 1) + (-1)(z - 4)}{\sqrt{1 + 4 + 1}(x - 2)^2 + (y + 1)^2 + (z - 4)^2} = \frac{4}{\sqrt{6}\sqrt{(x - 2)^2 + (y + 1)^2 + (z - 4)^2}}
\]

\[
\therefore \frac{1}{\sqrt{6}} = \frac{x + 2y - z + 4}{\sqrt{(x - 2)^2 + (y + 1)^2 + (z - 4)^2}}
\]

Hence, the required equation of the right circular cone is,

\[
16[(x - 2)^2 + (y + 1)^2 + (z - 4)^2] = (x + 2y - z + 4)^2
\]
i.e.

\[
15x^2 + 12y^2 + 15z^2 - 4yz - 2xz = 4xy - 80x + 16y - 120z + 320 = 0
\]

9.5 Cylinders

**Definition 9.3** A surface generated by a straight line which always remains parallel to the given fixed line and which intersects to the given curve is called a **cylinder**. The straight lines which generate the cylinder is called as the **generators** of the cylinder and the given curve is called as the **guiding curve**.

9.6 Equation of a cylinder

To find the equation of the cylinder whose generator intersects the conic, \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\), and are parallel to the line, \(\frac{x}{l} = \frac{y}{m} = \frac{z}{n}\).

Let \(P(x, y, z)\) be any point on the cylinder. Then generator through \(P\) is parallel to the line \(\frac{x}{l} = \frac{y}{m} = \frac{z}{n}\). Hence the d.r.s of the generator are \(l, m, n\). The equation of the generator through \(P\) is,

\[
\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}
\]

The point of intersection of the generator and the plane \(z = 0\) is given by,

\[
\frac{x - x_1}{l} = \frac{y - y_1}{m} = 0 - z_1.
\]

\[
\therefore \frac{x - x_1}{l} = -\frac{z_1}{n} \quad \text{and} \quad \frac{y - y_1}{m} = -\frac{z_1}{n} \quad \text{or} \quad x = x_1 - \frac{z_1}{n}, \quad y = y_1 - \frac{mz_1}{n}.
\]

Therefore the coordinates of the point of intersection of the generator and the plane \(z = 0\) are, \((x_1 - \frac{z_1}{n}, y_1 - \frac{mz_1}{n}, 0)\). But this point lies on the guiding curve \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\).

\[
\therefore a(x_1 - \frac{z_1}{n})^2 + 2h(x_1 - \frac{z_1}{n})(y_1 - \frac{mz_1}{n}) + b(y_1 - \frac{mz_1}{n})^2
\]

\[
+ 2g(x_1 - \frac{z_1}{n}) + 2f(y_1 - \frac{mz_1}{n}) + c = 0
\]

Hence, the locus of \(P\) is,

\[
a(x - \frac{z}{n})^2 + 2h(x - \frac{z}{n})(y - \frac{mz}{n}) + b(y - \frac{mz}{n})^2
\]

\[
+ 2g(x - \frac{z}{n}) + 2f(y - \frac{mz}{n}) + c = 0
\]

which when simplified, gives the equation of the cylinder as,

\[
a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2
\]

\[
+ 2gn(nx - lz) + 2fn(ny - mz) + cn^2 = 0
\]
Example 9.3 Find the equation of a cylinder whose generators are parallel to the line \( \frac{x}{2} = \frac{y}{1} = \frac{z}{3} \) and whose guiding curve is the ellipse \( x^2 + 2y^2 = 1 \) and \( z = 0 \).

**Solution.** Let \( P(x_1, y_1, z_1) \) be any point on the cylinder. Then the generator through \( P \) is parallel to the line, \( \frac{x}{2} = \frac{y}{1} = \frac{z}{3} \). Therefore the equation of the generator through \( P \) are,

\[
\frac{x - x_1}{2} = \frac{y - y_1}{1} = \frac{z - z_1}{3} = t \text{(say)}
\]

Coordinates of any point on this line are \((x_1 + 2t, y_1 + t, z_1 + 3t)\). For some \( t \), this point lies on guiding curve \( x^2 + 2y^2 = 1, z = 0 \).

\[\therefore (x_1 + 2t)^2 + 2(y_1 + t)^2 = 1 \quad (9.11)\]

also,

\[z_1 + 3t = 0, \therefore t = -\frac{z_1}{3}.
\]

Substituting the value of \( t \) in \( (9.11) \), we get

\[
(x_1 - \frac{2z_1}{3})^2 + 2(y_1 - \frac{z_1}{3})^2 = 1
\]

\[\therefore (3x_1 - 2z_1)^2 + 2(3y_1 - z_1)^2 = 9
\]

Hence, locus of \( P \) is, \((3x - 2z)^2 + 2(3y - z)^2 = 9\).

i.e. \( 9x^2 + 18y^2 + 6y^2 - 12xz - 12yz - 9 = 0 \),

i.e. \( 3x^2 + 6y^2 + 2y^2 - 4xz - 4yz - 3 = 0 \)

### 9.7 Right circular cylinder

**Definition 9.4** A cylinder is called a **right circular cylinder** if its guiding curve is a circle and its generators are lines perpendicular to the plane containing the circle. The normal to the plane of the guiding circle passing through its centre is called as the **axis of the cylinder**. If we take a section of the cylinder by a plane perpendicular to the axis of the cylinder, then this section will be a circle. The radius of this circle is called as the **radius of the cylinder**.

### 9.7.1 Equation of a right circular cylinder

To find the equation of the right circular cylinder whose axis is the line \( L : \frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \) and radius is \( r \). The point \( A(\alpha, \beta, \gamma) \) lies on the line \( L \) whose d.r.s. are \( l, m, n \). Let \( P(x, y, z) \) be a point on the cylinder see Fig. 9.4

Draw \( PM \) perpendicular to the axis of the cylinder. Then \( PM = r \). Now,

\[
AP^2 = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2
\]

\[
MA = \text{projection of } AP \text{ on the axis}
\]

\[\therefore MA = \frac{l(x - \alpha) + m(y - \beta) + n(z - \gamma)}{\sqrt{l^2 + m^2 + n^2}}
\]

Now, from the right angled \( \triangle AMP \), we get

\[
AP^2 - MA^2 = PM^2;
\]

\[
(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2
\]

\[
- \left( \frac{l(x - \alpha) + m(y - \beta) + n(z - \gamma)}{\sqrt{l^2 + m^2 + n^2}} \right)^2 = r^2
\]

which is the required equation of the cylinder.

**Example 9.4** Find the equation of the circular cylinder of radius 3 and axis passing through \((2, -1, 3)\) and having direction cosines proportional to \(1, 2, -2\).

**Solution.** Let \( A(2, -1, 3) \) and \( P(x, y, z) \) be any point on the cylinder. Draw \( PM \) perpendicular to the axis of the cylinder. Then \( PM = 3 \). By the distance formula,

\[
AP^2 = (x - 2)^2 + (y + 1)^2 + (z - 3)^2
\]

\[
MA = \text{projection of } AP \text{ on the axis}
\]

\[
\therefore MA = \frac{1(x - 2) + 2(y + 1) - 2(z - 3)}{\sqrt{1^2 + 2^2 + (-2)^2}}
\]

\[
MA = \frac{x + 2y - 2y + 6}{3}
\]
Now, from the right angled triangle $\triangle AMP$, we have $AP^2 - MA^2 = 9$.

Substituting the value of $AP$ and $MA$, we get

$$8x^2 + 5y^2 + 5z^2 + 8yz + 4zx - 4xy - 48x - 6y - 30z + 9 = 0,$$

which is the required equation of the cylinder.

### 9.8 Illustrative Examples

**Example 9.5** Find the general equation of the quadratic cone with the vertex at the origin and passing through the three coordinate axes.

**Solution.** The vertex of the cone is the origin. Hence its equation is a homogeneous equation of degree 2 in $x, y, z$. Let the equation of the cone be,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad (9.12)$$

The cone given by (9.12) passes through the three coordinate axes. Therefore $x, y$ and $z$ axes are the generators of the cone. The direction ratios of $x, y$ and $z$ axes satisfy the equation (9.12). d.r.s of the $X$ - axis are 1, 0, 0; d.r.s of the $Y$ - axis are 0, 1, 0 and d.r.s of the $Z$ - axis are 0, 0, 1. From these conditions we get $a = 0, b = 0$ and $c = 0$. Substituting $a = 0, b = 0$ and $c = 0$ in (9.12), we get the general equation of the cone which passes through the three co-ordinate axes as

$$2fyz + 2gzx + 2hxy = 0 \text{ i.e. } fyz + gzx + hxy = 0.$$

**Example 9.6** Find the equation of the cone which passes through the axes of co-ordinates and contains the points $(1, 1, 1)$ and $(1, -2, 1)$.

**Solution.** As the cone passes through the three co-ordinate axes, the vertex of the cone is at the origin. Hence the equation of the cone is of the form

$$fy + gz + hx = 0 \quad (9.13)$$

The cone passes through the points $(1, 1, 1)$ and $(1, -2, 1)$. Co-ordinates of these two points satisfy the equation (9.13).

$$f + g + h = 0 \quad \text{and} \quad 3f - g - 2h = 0$$

Solving the equations for $f, g$ and $h$, we get

$$\frac{f}{-1} = \frac{g}{4} = \frac{h}{-3} = k \text{ (say)}$$

$$\therefore f = -k, \quad g = 4k, \quad h = -3k.$$ Substituting the values of $f, g$ and $h$ in (9.13) we get the equation of the required cone as,

$$-kyz + 4kzx - 3kxy = 0 \text{ i.e. } yz - 4zx + 3xy = 0.$$

**Example 9.7** Find the equation of the cone passing through the co-ordinate axes and having the lines \( x \quad 1 = \quad y \quad -2 = \quad z \quad 3 \) and \( x \quad 3 = \quad y \quad -1 = \quad z \quad -1 \) as generators.

**Solution.** As the cone passes through the three co-ordinate axes, the vertex of the cone is at the origin. Hence the equation of the cone is of the form

$$fy + gz + hx = 0 \quad (9.14)$$
9.8. Illustrative Examples

Given that the lines \( \frac{x}{2} = \frac{y}{4} = \frac{z}{2} \) and \( \frac{x}{2} = \frac{w}{2} = \frac{z}{2} \) are the generators of the cone, thus d.r.s of the lines satisfy the equation (8.8.3)

\[
\therefore \quad f(-2)(3) + g(3)(1) + h(1)(-2) = 0
\]

and \( f(-1)(-1) + g(-1)(3) + h(3)(-1) = 0 \)

\[
\therefore \quad -6f + 3g - 2h = 0 \quad \text{and} \quad f - 3g - 3h = 0.
\]

Solving these two equations for \( f, g \) and \( h \) we get

\[
\frac{f}{-1} = \frac{g}{4} = \frac{h}{-3} = k \text{ (say)}
\]

\[
\therefore \quad f = -15k, \quad g = -20k, \quad h = 15k.
\]

Substituting the values of \( f, g \) and \( h \) in (8.8.3), we get the equation of the required cone as

\[
-15k yz - 20k zx + 15k xy = 0 \quad \text{i.e.} \quad 3yz + 4zx + 3xy = 0.
\]

Example 9.8 Show that the line \( \frac{x}{2} = \frac{w}{4} = \frac{z}{2} \) is a generator of the cone \( x^2 + y^2 + z^2 + 4xy - xz = 0 \).

Solution. The equation of the cone is,

\[
x^2 + y^2 + z^2 + 4xy - xz = 0 \quad (9.15)
\]

which is a homogenous equation. Hence the vertex of the cone is the origin. If we show that d.r.s of the line \( \frac{x}{2} = \frac{w}{4} = \frac{z}{2} \) satisfy the equation (9.15), then we can say that the given line is a generator of the cone given by the equation (9.15). d.r.s of the line \( \frac{x}{2} = \frac{w}{4} = \frac{z}{2} \) are \( 2, -1, 3 \).

Substituting these values in \( L.H.S \) of (9.15) we get,

\[
2^2 + (-1)^2 + 3^2 + 4(2)(-1) - (2)(3).
\]

Which is equal to 0. Thus, the line \( \frac{x}{2} = \frac{w}{4} = \frac{z}{2} \) is the generator of the given cone.

Example 9.9 Show that the equation \( 4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0 \) represents a cone with vertex at the point \((-1, -2, -3)\).

Solution.

\[
4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0 \quad (9.16)
\]

Shift the origin to the point \((-1, -2, -3)\). Let \((x, y, z)\) and \((x', y', z')\) be respectively the old and new co-ordinates of the points on the cone. Then \( x = x' - 1, \quad y = y' - 1, \quad z = z' - 1 \) Substituting in to the equation (9.16) we get

\[
4(x' - 1)^2 - (y' - 2)^2 + 2(z' - 3)^2 + 2(x' - 1)(y' - 2) - 3(y' - 2)(z' - 3) + 12(x' - 1) - 11(y' - 2) + 6(z' - 3) + 4 = 0
\]

On simplification gives

\[
4x'^2 - y'^2 + 2z'^2 + 2x'y' - 3y'z' = 0 \quad (9.17)
\]

The equation (9.17) is a homogeneous equation in \( x', y', z' \). Hence the equation (9.17) represents a cone with vertex at origin in the new co-ordinate system. Thus the equation (9.16) represents a cone with vertex at \((-1, -2, -3)\).

Example 9.10 Find the equation of the cone whose vertex is at origin and the guiding curve is a circle \( y^2 + z^2 = 16, \quad x = 2 \). Show that section of the cone by the plane \( z = 1 \) is a hyperbola.

Solution. Since the vertex of the cone is at origin, the equation of the cone is a homogeneous equation in \( x, y, z \). Consider the equation of the guiding curve \( y^2 + z^2 = 16, \quad x = 2 \). We make one of the equations homogeneous with the help of the other. We make \( y^2 + z^2 = 16 \) homogeneous with the help of \( x = 2 \)

\[
\therefore \quad y^2 + z^2 = 16 \times 1^2 \quad \therefore \quad y^2 + z^2 = 16 \times \left( \frac{x}{2} \right)^2, \quad \text{since} \quad \frac{x}{2} = 1
\]

Hence the equation of the cone is \( 4x^2 - y^2 - z^2 = 0 \). Now the section of the cone by the plane \( z = 1 \) is,

\[
4x^2 - y^2 - 1^2 = 0, \quad z = 1 \text{ i.e.} \quad 4x^2 - y^2 = 1, \quad z = 1
\]
which is a hyperbola.

**Example 9.11** Find the equation of the cone with its vertex at the origin and whose guiding curve is given by \( x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0 \), \( x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0 \).

**Solution.** Since the vertex of the cone is at origin, the equation of the cone is a homogeneous equation in \( x, y, z \). The equation of the cone is obtained by making the equation
\[
x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0 \tag{9.18}
\]
homogeneous with the help of the equation
\[
x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0 \tag{9.19}
\]
Subtracting the equation (9.18) from the equation (9.19) we get
\[
4x + 2y + 2z - 8 = 0 \quad \text{i.e.} \quad \frac{2x + y + z}{4} = 1 \tag{9.20}
\]
using (9.20) we make the equation (9.17) homogeneous as follows,
\[
x^2 + y^2 + z^2 - 2x \left( \frac{2x + y + z}{4} \right)
+ 2y \left( \frac{2x + y + z}{4} \right) + 4z \left( \frac{2x + y + z}{4} \right) - 3 \left( \frac{2x + y + z}{4} \right) = 0
\]
Simplification yields,
\[
12x^2 - 21y^2 - 29z^2 - 18yz - 12zx + 4xy = 0
\]
which is the required equation of the cone.

**Example 9.12** Find the equation of the right circular cone which passes through the point \((1, -2, 3)\) whose vertex is at \((2, -3, 5)\) and whose axis makes equal angles with co-ordinate axes.

**Solution.** We are given that axis of the right circular cone makes equal angles with co-ordinate axes. Therefore the \(d.r.s\) of the axis are 1, 1, 1. Let \(A(1, -2, 3)\) and \(V(2, -3, 5)\). Therefore the \(d.r.s\) of \(VA\) are 2, -1, -3 + 2, 5 - 3 i.e. 1, -1 and 2 respectively. Let \(\theta\) be the semi vertical angle. The \(\theta\) is the angle between the axis of the cone and \(VA\)
\[
\therefore \cos \theta = \frac{1(1) + 1(-1) + 1(2)}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{(-1)^2 + 2^2}} = \frac{2}{\sqrt{3} \sqrt{6}} \tag{9.21}
\]
Let \(P(x, y, z)\) be a point on the right circular cone. Then \(VP\) is a generator and its \(d.r.s\) are \(x - 2, y + 3, z - 5\). Then \(\theta\) is the angle between \(VP\) and the axis
\[
\therefore \cos \theta = \frac{1(x - 2) + 1(y + 3) + 1(z - 5)}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{(x - 2)^2 + (y - 3)^2 + (z - 5)^2}} \tag{9.22}
\]
From (9.21) and (9.22) we get
\[
\frac{2}{\sqrt{3} \sqrt{6}} = \frac{x + y + z - 4}{\sqrt{3} \sqrt{3(x - 2)^2 + (y - 3)^2 + (z - 5)^2}}
\]
i.e.
\[
2[(x - 2)^2 + (y - 3)^2 + (z - 5)^2]^2 = 3(x + y + z - 4)^2,
\]
which is the required equation of the right circular cone.

**Example 9.13** Find the equation of the right circular cylinder of radius 2, whose axis passes through \((1, 2, 3)\) and has \(d.c.s\) proportional to 2, -3, 6.

**Solution.** Let \(A(1, 2, 3)\) and \(P(x, y, z)\) be a point on the cylinder. Draw \(PM\) perpendicular to the axis of the cylinder. Then \(PM\) is the radius of the cylinder so \(PM = 2\). By Distance formula, \(AP^2 = (x - 1)^2 + (y - 2)^2 + (z - 3)^2\). Let \(MA\) be the projection of \(AP\) on the axis. \(\therefore MA = \frac{2(x - 1) - 3(y - 2) + 6(z - 3)}{\sqrt{2^2 + (-3)^2 + 6^2}}\) as \(d.r.s\) of the axis are 2, -3, 6. Thus, \(MA = \frac{2x - 3y + 6z - 14}{7}\)

Now from the right angled \(\triangle MPA\), we get \(AP^2 - MA^2 = 9\)
\[
[(x - 1)^2 + (y - 2)^2 + (z - 3)^2] - \left(\frac{2x - 3y + 6z - 14}{7}\right)^2 = 9
\]
Simplification of the equation gives the required equation of the right circular cylinder as
\[
45x^2 + 40y^2 + 13z^2 + 36yz - 24zx + 12xy - 42x - 280y - 126z + 294 = 0.
\]
9.9 Exercise

1. Find the equation of a cone whose vertex is at $(-1, 1, 2)$ and guiding curve is $3x^2 - y^2 = 1; z = 0$.

2. Find the equation of a cone with vertex at the origin and containing the curve $x^2 + y^2 = 4; z = 5$.

3. Find the equation of a cone whose vertex is at $(1, 1, 3)$ and passing through $4x^2 + z^2 = 1; y = 4$.

4. The axis of a right circular cone with vertex at the origin makes equal angles with the coordinate axes. If the cone passes through the line drawn from the origin with direction ratios $1, -2, 2$, find the equation of the cone.

5. Find the equation of the cylinder whose generators are parallel to the line $6x = -3y = 2z$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1; z = 3$.

6. Lines are drawn parallel to the line $x = \frac{y-1}{2} = \frac{z-5}{3}$ through the points on the circle $x^2 + y^2 = a^2$ in $ZOX$-plane. Find the equation of the surface so formed.

7. Find the equation of the right circular cylinder of radius 2 and having as axis the line $\frac{x-3}{1} = \frac{y-2}{4} = \frac{z+3}{2}$.

8. Find the equation of the right circular cone having $P(2, -3, 5)$ as a vertex; axis $PQ$ which makes equal angles with coordinate axes and the semi vertical angle is $30^\circ$.

9. Show that $x^2 + 2y^2 + 2 - 4yz - 6zx - 2x + 8y - 2 + 9 = 0$ represents a cone with vertex at $(1, -2, 0)$.

10. Find the equation of a cone with vertex the origin and base a circle in the plane $z = 12$ with centre $(13, 0, 12)$ and radius 5. Also show that the section of any plane parallel to $x = 0$ is a circle.

11. Find the equation of the right circular cylinder of radius 2 whose axis passes through $(1, 2, 3)$ and has d.r.s. $2, -3, 6$.

12. Find the equation of a cone whose vertex is at the origin and direction ratios of whose generators satisfy the equation $3l^2 - 2m^2 + 5n^2 = 0$.

13. The equation of a cone is $x^2 + 2y^2 + z^2 - 2yz + 2z - 3xy = 0$. Test whether the following lines are generators of the cone.
   (a) $x = y = z$ (b) $x = y = z$ (c) $\frac{y}{x} = \frac{y}{z} = \frac{z}{x}$ (d) $\frac{y}{x} = \frac{y}{z} = \frac{z}{y}$.

14. Find the equation of a cone with vertex at the origin and which passes through the curve $x^2 + y^2 + z^2 + x - 2y + 3z = 4; x^2 + y^2 + z^2 + 2x - 3y + 4z = 5$.

15. Find the equation of the right circular cylinder of radius 2 whose axis lies along the line $\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{5}$.

16. Obtain the equation of the right circular cylinder whose guiding curve is the circle $x^2 + y^2 + z^2 - 9 = 0; x - y + z - 3 = 0$.

17. Lines are drawn through the origin having direction ratios $1, 2, 3; 2, 3, 6$; and $3, 4, 12$. Show that the axis of the right circular cone through them has d.c.s. $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{6}}$ and the semi vertical angle of the cone is $-\ldots$. Also obtain the equation of the cone.

18. Show that the equation $2y^2 - 8yz - 4xz - 8xy + 6x - 4y - 2z + 5 = 0$ represents a cone whose vertex is $(-\frac{7}{6}, \frac{1}{3}, \frac{5}{6})$.

19. Determine the equation of the right circular cone having vertex at $(2, 3, 1)$, axis parallel to the line $2x = -y = -2z$ and one of its generators having d.r.s. $1, 1, 1$.

20. Find the equation of the right circular cone generated by the lines drawn from the origin to cut the circle through the points $(1, 2, 2)$, $(2, 1, -2)$ and $(2, -2, 1)$.

21. Find the equation of the cone with vertex at the origin and containing the curve $ax^2 + by^2 = 2z; lx + my + nz = p$. 
22. Obtain the equation of the right circular cone which is generated by revolving the line whose equations are \(3x - y + z = 1; 5x + y + 3z + 1 = 0\) about the y-axis.

23. Find the equation of the cone which passes through the coordinate axes and has two generators having direction ratios 1, 2, 2 and -2, -2, 1.

24. Obtain the equation of the cone which passes through the coordinate axes and has the lines \(x^2 = y^1 = z^3\) and \(x - 3 = y - 1 = z - 2\) as its generators.

9.10 Answers

(1) \(12x^2 - 4y^2 + z^2 + 4yz + 12zx + 4z - 4 = 0\).
(2) \(25(x^2 + y^2) - 4z^2 = 0\).
(3) \(12x^2 + 4y^2 + 3z^2 + 6yz + 8xy - 32x - 34y - 24z + 69 = 0\).
(4) \(4x^2 + 4y^2 + 4z^2 + 9yz + 9zx + 9xy = 0\).
(5) \(3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0\).
(6) \((mx - ly)^2 + (nz - my)^2 = m^2a^2\).
(7) \(5x^2 + 8y^2 + 5z^2 - 4yz - 8zx - 4xy + 22x - 16y - 14z + 26 = 0\).
(8) \(5x^2 + 5y^2 + 5z^2 - 8yz - 8zx - 8xy + 8x + 86y + 278 = 0\).
(10) \(6x^2 + 6y^2 + 6z^2 - 13xz = 0\).
(11) \(9(2y + z - 7)^2 + 4(z - 3x)^2 + (3x + 2y - 7)^2 = 196\).
(12) \(3x^2 - 2y^2 + 5z^2 = 0\).
(13) \((b)\) and \((c)\) are generators; \((a)\) and \((d)\) are not generators.
(14) \(x^2 + y^2 - z^2 = 0\).
(15) \(2x^2 + y^2 - 3yz + 4zx - 5xy = 0\).
(16) \(26x^2 + 29y^2 + 5z^2 + 10yz - 24zx - 4xy + 150y + 30z + 75 = 0\).
(17) \(x^2 + y^2 + z^2 - zx + xy = 0\). (18) \(yz - zx - xy = 0\).
(20) \(x^2 - 8y^2 - z^2 - 12yz + 6zx + 12xy - 46x + 38y + 22z - 19 = 0\).
(21) \(8x^2 - 4y^2 - 4z^2 + 9yz + 5zx + 5xy = 0\).
(22) \(apx^2 + bpy^2 - 2nz^2 - 2myz - 2lxz = 0\).
(23) \(x^2 - 5y^2 + z^2 - 10y - 5 = 0\).
(24) \(yz - zx - xy = 0\). (25) \(6yz - zx - 6xy = 0\).