Chapter 9

Cones and Cylinders

9.1 Cone

Definition 9.1 A surface generated by a straight line passing through a fixed point and intersecting a given curve is called as $a\ cone$.
The fixed point is called the $vertex\ of\ the\ cone$ and the given curve is called the $guiding\ curve$. A line which generates the cone is called $a\ generator$.

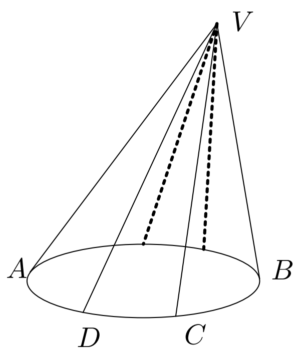

Figure 9.1: Cone with a guiding curve

The surface in the Fig. 9.1 is a cone with vertex V, The line VA as a generator. The lines VB, VC are also generators. In fact every line joining V to any point of the guiding curve is a generator of the cone.

Remark 9.1 If the guiding curve is a plane curve of degree n, then the equation of the cone is also of degree n and we call it $a\ cone\ of\ order\ n$.

In this chapter, we study only quadratic cones; i.e. cones having its equation of second degree in x, y and z.

9.2 Equation of a cone

To find the equation of a cone with vertex $V(\alpha, \beta, \gamma)$ and whose guiding curve is the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0. \qquad (9.1)$$

We have to find the locus of points on lines which pass through the vertex $V(\alpha, \beta, \gamma)$ and intersects the given guiding curve. Observe that the equations of any line passing through the vertex $V(\alpha, \beta, \gamma)$ and having direction ratios l, m, n is given by,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad (9.2)$$

Any point on the equation (9.2) has the coordinates, $(\alpha+lt, \beta+mt, \gamma+nt)$, where $t \in \mathbb{R}$. If the line (9.2) intersects the guiding curve given by (9.1), then we have,

$$\begin{aligned} a(\alpha+lt)^2 + 2h(\alpha+lt)(\beta+mt) + b(\beta+mt)^2 + 2g(\alpha+lt) \\ +2f(\beta+mt) + c = 0, (\gamma+nt) = 0 \end{aligned} \qquad (9.3)$$

Eliminating t, l, m and n between (9.2) and (9.3), we get

$$\begin{aligned} a(\alpha z+\gamma x)^2 + 2h(\alpha z+\gamma x)(\beta z+\gamma y) + b(\beta z+\gamma y)^2 \\ +2g(\alpha z+\gamma x)(z-y) + 2f(\beta z+\gamma y)(z-y) + c(z-y)^2 &= 0. \end{aligned} \qquad (9.4)$$

This is the required equation of the cone.

Remark 9.2 *From the equation* (9.4) *it can be seen that the equation of a quadratic cone is of second degree in x, y and z; i.e.,*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad (9.5)$$

Example 9.1 *Find the equation of a cone with vertex at the point (3,1,2) and guiding curve is* $2x^2 + 3y^2 = 1, z = 0$.

Solution The vertex is $V(3,1,2)$. Let a, b, c be the direction ratios of a generator of the cone. Then the equations of generator are,

$$\frac{x-3}{a} = \frac{y-1}{b} = \frac{z-2}{c} = t(say) \tag{9.6}$$

The coordinates of any point on the generator are $(3+at, 1+bt, 2+ct)$. For some $t \in \mathbb{R}$, $(3+at, 1+bt, 2+ct)$ lies on the guiding curve. Therefore $2(3+at)^2 + 3(1+bt)^2 = 1$ and $2 + ct = 0$. Thus, $t = \frac{-2}{c}$. From this we get,

$$2\left(3 - \frac{2a}{c}\right)^2 + 3\left(1 - \frac{2b}{c}\right)^2 = 1$$

From (9.6), we obtain $2[3 - 2(\frac{x-3}{z-2})]^2 + 3[1 - 2(\frac{y-1}{z-2})]^2 = 1$ so $2[3(z-2) - 2(x-3)]^2 + 3[1(z-2) - 2(y-1)]^2 = (z-2)^2$.

After simplification, the required equation of the cone is,

$$2x^2 + 3y^2 + 5z^2 - 3yz - 6xz + z - 1 = 0.$$

9.3 Cone with vertex at the origin

Recall that the general equation of second degree equation is,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

If any one of the constants u, v, w and d is non-zero, then the equation is non-homogeneous in x, y and z. If each of the constants u, v, w and d, is zero, then the resulting equation, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, is a homogeneous second degree equation in x, y and z.

Theorem 9.1 The equation of a cone with vertex at the origin is a homogeneous second degree equation.

Proof. Let the equation of a quadratic cone with vertex at the origin be,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \tag{9.7}$$

We will show that $u = v = w = d = 0$. Let $P(x', y', z')$ be any point on the cone given by (9.7). Since $O(0,0,0)$ is the vertex, the line OP is a generator of the cone. The equation of the line OP is given by,

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = t(say)$$

Thus the coordinates of any point on the line OP are (tx', ty', tz'). Since OP is a generator, these coordinates satisfy the equation (9.7),

$\therefore a(tx')^2 + b(ty')^2 + c(tz')^2 + 2f(ty')(tz')$
$+ 2g(tz')(tx') + 2h(tx')(ty') + 2u(tx') + 2v(ty') + 2w(tz') + d = 0.$

This equation can be written as,

$$\begin{aligned}(ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y')t^2 \\ +2(ux' + 2vy' + 2wz')t + d = 0\end{aligned} \tag{9.8}$$

Observe that (9.8) is a quadratic equation for $\forall\, t \in \mathbb{R}$. This is possible only if each of the coefficient is 0. Therefore,

$$\begin{aligned} ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' &= 0, \\ ux' + 2vy' + 2wz' = 0 \text{ and } \quad d = 0. \end{aligned}$$

We now claim that each of the constants u, v and w are zero. For if not i.e. if at least one of the constants is not zero, then the equation $ux + vy + wz = 0$ would represent a plane and the point (x', y', z') lies on the plane. As the point (x', y', z') lies on the surface (9.7) , it would mean that the surface (9.7) is a plane. This is impossible as it represents a cone. Hence $u = 0, v = 0, w = 0$ and we also have $d = 0$. So that the equation of a cone with vertex at the origin is of the form,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

which is a homogeneous in x, y and z. ∎

Note that the converse of this theorem is also true.

Theorem 9.2 Every second degree homogeneous equation in x, y, z represents a cone with vertex at the origin.
Proof. Consider the homogeneous equation of second degree in x, y, z

$$i.e. \quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \tag{9.9}$$

The coordinates of the origin satisfy the equation (9.10). Thus, the origin O lies on the surface given by (9.9). Let $P(x', y', z')$ be any other point on the surface given by (9.10).

$$\therefore ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0 \tag{9.10}$$

Now we show that the line OP lies on the surface given by (9.9). The equation of the line OP is,

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = t(say)$$

The coordinates of any point on the line OP are (tx', ty', tz'). Thus substituting these coordinates in $L.H.S.$ of (9.9) we have,

$$\begin{aligned}
&a(tx')^2 + b(ty')^2 + c(tz')^2 + 2f(ty')(tz') + 2g(tz')(tx') + 2h(tx')(ty') \\
=\ & t^2(ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y') \\
=\ & t^2(0) \quad \ldots (9.9) \\
=\ & 0.
\end{aligned}$$

Hence, the point with coordinates (tx', ty', tz') satisfies (9.9). Therefore any point on the line OP lies on the surface given by (9.9). Thus (9.9) is the surface generates by OP. Therefore the equation (9.9) represents a cone with vertex at the origin.

Remark 9.3 If the line with equation, $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator of a cone with vertex at the origin, then the direction ratios l, m, n satisfies the equation of the cone.

Let the equation of the cone be,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

The coordinates of any point on the line are (lt, mt, nt), where $t \in \mathbb{R}$. These coordinates satisfy the equation.

$$\begin{aligned}
\therefore \quad & a(lt)^2 + b(mt)^2 + c(nt)^2 + 2f(mt)(nt) \\
& +2g(nt)(lt) + 2h(lt)(mt) = 0 \\
\therefore \quad & t^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) = 0.
\end{aligned}$$

This equation holds of all values of t.

$$\therefore al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0.$$

Thus, it follows that the direction ratios of a generator satisfy the equation of the cone.

Also, it is easy to see that if the direction ratios l, m, n of a generator of a cone with vertex at the origin satisfy the equation $\phi(l, m, n) = 0$, then the equation of the cone is $\phi(x, y, z) = 0$. $\blacksquare$

Remark 9.4 The general equation of a quadratic cone with vertex at the origin is,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Let the origin be shifted to the point $V(\alpha, \beta, \gamma)$. In the new coordinate system the above equation becomes,

$$\begin{aligned}
& a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 \\
+ \quad & 2f(y-\beta)(z-\gamma) + 2g(z-\gamma)(x-\alpha) + 2h(x-\alpha)(y-\beta) = 0.
\end{aligned}$$

This is a general equation of a quadratic cone with vertex at (α, β, γ).

9.4 The Right circular Cone

Definition 9.2 A right circular cone is a surface generated by a straight line passing through a fixed point and making a constant angle θ with a fixed straight line passing through the given point. The fixed point is called as $the\ vertex$ of the right circular cone, the fixed straight line is called as $the\ axis$ of the cone and the constant angle θ is called as $the\ semi-vertical\ angle$ of the cone.

Remark 9.5 Every section of a right circular cone by a plane perpendicular to its axis is a circle.

Let θ be the semi - vertical angle of the cone and α be a plane perpendicular to the axis VN see Fig. 9.2 of the cone. We show that the section of the cone by the plane α is a circle.

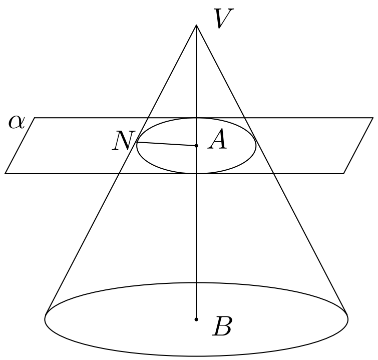

Figure 9.2: Section of a cone with a plane

Let P be any point of the section of the cone by the plane α. Let A be the point of intersection of the axis VN and the plane α. Then AP is perpendicular to VA. In the right angled triangle VAP, $\tan\theta = \frac{AP}{AV}$. $\therefore AP = AV\tan\theta$. Since, AV and $\tan\theta$ are constant. it follows that AP is constant for all points P on the section of the cone by the plane α. Thus the section is a circle.

9.4.1 Equation of a right circular cone

To find the equation of a right circular cone with a vertex $V(\alpha.\beta,\gamma)$ and whose axis is the line, $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, with a semi-vertical angle θ. Let VN be the axis of the cone and let $P(x,y,z)$ be any point on the cone. The direction rations of the generator VP are $x-\alpha, y-\beta, z-\gamma$.

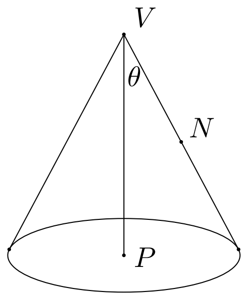

Figure 9.3:

Let P be any point of the section of the cone by the plane α. Let A be the point of intersection of the axis VN and the plane α. Then AP is perpendicular to VA. In the right angled triangle VAP, $\tan\theta = \frac{AP}{AV}$. $\therefore AP = AV\tan\theta$. Since, AV and $\tan\theta$ are constant. it follows that AP is constant for all points P on the section of the cone by the plane α. Thus the section is a circle.

The angle between VP and VN is $\theta(see\ Fig.\ 9.3)$. As the direction ratios of the axis VN are l,m,n, we have,

$$\cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Therefore, the equation of the right circular cone is,

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+m^2+n^2)\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]\cos^2\theta$$

The following example will make the above proof more clear.

Example 9.2 Find the equation of the right circular cone with vertex at $(2,-1,4)$, the line $\dfrac{x-2}{1}=\dfrac{y+1}{2}=\dfrac{z-4}{-1}$ as the axis and semi-vertical angle $\cos^{-1}(4/\sqrt{6})$.

Solution. Let $P(x,y,z)$ be any point on the cone with vertex $V(2,-1,4)$. Then the direction ratios of a generator VP are $x-2, y+1, z-4$. The direction ratios of the axis $\dfrac{x-2}{1}=\dfrac{y+1}{2}=\dfrac{z-4}{-1}$ are $1,2,-1$. Let θ be the semi-vertical angle. Hence, $\cos\theta=\frac{4}{\sqrt{6}}$. From $Remark$, we have,

$$\cos\theta = \frac{1(x-2)+2(y+1)+(-1)(z-4)}{\sqrt{1+4+1}\sqrt{(x-2)^2+(y+1)^2+(z-4)^2}}$$
$$\therefore \quad \frac{4}{\sqrt{6}} = \frac{x+2y-z+4}{\sqrt{6}\sqrt{(x-2)^2+(y+1)^2+(z-4)^2}}$$

Hence, the required equation of the right circular cone is,

$$16\left[(x-2)^2+(y+1)^2+(z-4)^2\right] = (x+2y-z+4)^2$$

i.e.

$$15x^2+12y^2+15z^2-4yz-2zx=4xy-80x+16y-120z+320=0$$

9.5 Cylinders

Definition 9.3 A surface generated by a straight line which always remains parallel to the given fixed line and which intersects to the given curve is called $a\ cylinder$. The straight lines which generate the cylinder is called as *the generators* of the cylinder and the given curve is called as *the guiding curve.*

9.6 Equation of a cylinder

To find the equation of the cylinder whose generator intersects the conic, $ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0$ and are parallel to the line, $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$.

Let $P(x,y,z)$ be any point on the cylinder. Then generator through P is parallel to the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$. Hence the $d.r.s$ of the generator are l,m,n. The equation of the generator through P is,

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

The point of intersection of the generator and the plane $z=0$ is given by,

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{0-z_1}{n}.$$
$$\therefore \quad \frac{x-x_1}{l}=\frac{-z_1}{n} \quad and \quad \frac{y-y_1}{l}=\frac{-z_1}{n}.$$
$$\therefore \quad x=x_1-\frac{lz_1}{n}, \quad y=y_1-\frac{mz_1}{n}.$$

Therefore the coordinates of the point of intersection of the generator and the plane $z=0$ are, $(x_1-\dfrac{lz_1}{n}, y_1-\dfrac{mz_1}{n}, 0)$. But this point lies on the guiding curve $ax^2+2hxy+by^2+2gx+2fy+c=0$.

$$\therefore \quad a(x_1-\frac{lz_1}{n})^2+2h(x_1-\frac{lz_1}{n})(y_1-\frac{mz_1}{n})+b(y_1-\frac{mz_1}{n})^2$$
$$+2g(x_1-\frac{lz_1}{n})+2f(y_1-\frac{mz_1}{n})+c=0$$

Hence, the locus of P is,

$$a(x-\frac{lz}{n})^2+2h(x-\frac{lz}{n})(y-\frac{mz}{n})+b(y-\frac{mz}{n})^2$$
$$+2g(x-\frac{lz}{n})+2f(y-\frac{mz}{n})+c=0$$

which when simplified, gives the equation of the cylinder as,

$$a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2$$
$$+2gn(nx-lz)+2fn(ny-mz)+cn^2=0$$

Example 9.3 Find the equation of a cylinder whose generators are parallel to the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2 = 1$ and $z = 0$.

Solution. Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then the generator through P is parallel to the line, $\frac{x}{2} = \frac{y}{1} = \frac{z}{3}$. Therefore the equation of the generator through P are,

$$\frac{x - x_1}{2} = \frac{y - y_1}{1} = \frac{z - z_1}{3} = t(say)$$

Coordinates of any point on this line are (x_1+2t, y_1+t, z_1+3t). For some t, this point lies on guiding curve $x^2 + 2y^2 = 1$, $z = 0$.

$$\therefore \quad (x_1 + 2t)^2 + 2(y_1 + t)^2 = 1 \tag{9.11}$$

also,

$$z_1 + 3t = 0, \therefore \quad t = \frac{-z_1}{3}.$$

Substituting the value of t in (9.11), we get

$$(x_1 - 2\frac{z_1}{3})^2 + 2(y_1 - \frac{z_1}{3})^2 = 1$$

$$\therefore (3x_1 - 2z_1)^2 + 2(3y_1 - z_1)^2 = 9$$

Hence, locus of P is, $(3x - 2z)^2 + 2(3y - z)^2 = 9$.

$$i.e. 9x^2 + 18y^2 + 6y^2 - 12xz - 12yz - 9 = 0,$$
$$i.e. 3x^2 + 6y^2 + 2y^2 - 4xz - 4yz - 3 = 0$$

9.7 Right circular cylinder

Definition 9.4 A cylinder is called a $right\ circular\ cylinder$ if its guiding curve is a circle and its generators are lines perpendicular to the plane containing the circle. The normal to the plane of the guiding circle passing through its centre is called as the $axis\ of\ the\ cylinder$. If we take a section of the cylinder by a plane perpendicular to the axis of the cylinder, then this section will be a circle. The radius of this circle is called as the $radius\ of\ the\ cylinder$.

9.7.1 Equation of a right circular cylinder

To find the equation of the right circular cylinder whose axis is the line $L : \frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$ and radius is r. The point $A(\alpha, \beta, \gamma)$ lies on the line L whose d.r.s. are l, m, n. Let $P(x, y, z)$ be a point on the cylinder $see\ Fig.\ 9.4$

Draw PM perpendicular to the axis of the cylinder. Then $PM = r$. Now,

$$\begin{aligned} AP^2 &= (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 \\ MA &= projection\ of\ AP\ on\ the\ axis \\ \therefore MA &= \frac{l(x - \alpha) + m(y - \beta) + n(z - \gamma)}{\sqrt{l^2 + m^2 + n^2}} \end{aligned}$$

Now, from the right angled $\triangle AMP$, we get

$$\begin{aligned} AP^2 - MA^2 &= PM^2; \\ (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 & \\ - \left(\frac{l(x - \alpha) + m(y - \beta) + n(z - \gamma)}{\sqrt{l^2 + m^2 + n^2}} \right)^2 &= r^2 \end{aligned}$$

which is the required equation of the cylinder.

Example 9.4 Find the equation of the circular cylinder of radius 3 and axis passing through $(2, -1, 3)$ and having direction cosines proportional to $1, 2, -2$.

Solution. Let $A(2, -1, 3)$ and $P(x, y, z)$ be any point on the cylinder. Draw PM perpendicular to the axis of the cylinder. Then $PM = 3$. By the distance formula,

$$\begin{aligned} AP^2 &= (x - 2)^2 + (y + 1)^2 + (z - 3)^2 \\ MA &= projection\ of\ AP\ on\ the\ axis \\ \therefore \quad MA &= \frac{1(x - 2) + 2(y + 1) - 2(z - 3)}{\sqrt{1^2 + 2^2 + (-2)^2}} \\ MA &= \frac{x + 2y - 2y + 6}{3} \end{aligned}$$

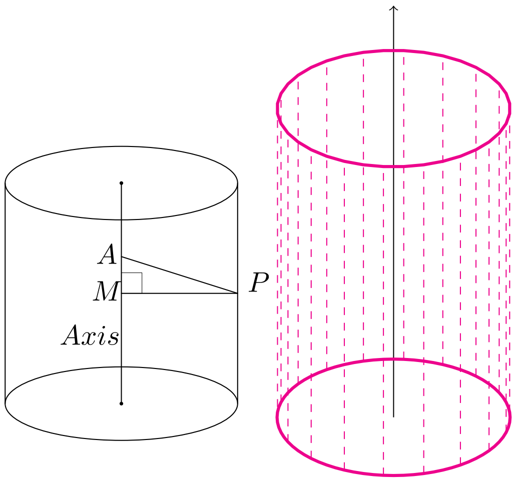

Figure 9.4:

Now, from the right angled triangle $\triangle AMP$, we have $AP^2 - MA^2 = 9$. Substituting the value of AP and MA, we get

$$8x^2 + 5y^2 + 5z^2 + 8yz + 4zx - 4xy - 48x - 6y - 30z + 9 = 0,$$

which is the required equation of the cylinder.

9.8 Illustrative Examples

Example 9.5 Find the general equation of the quadratic cone with the vertex at the origin and passing through the three coordinate axes.
Solution. The vertex of the cone is the origin.Hence its equation is a homogenous equation of degree 2 in x, y, z. Let the equation of the cone be,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \tag{9.12}$$

The cone given by (9.12) passes through the three co-ordinate axes. Therefore x, y and z axes are the generators of the cone. The direction

ratios of x, y and z axes satisfy the equation (9.12). $d.r.s$of the $X - axis$ are $1, 0, 0$; $d.r.s$ of the $Y - axis$ are $0, 1, 0$ and $d.r.s$ of the $Z - axis$ are $0, 0, 1$. From these conditions we get $a = 0, b = 0$ and $c = 0$. Substituting $a = 0,\ b = 0$ and $c = 0$ in (9.12), we get the general equation of the cone which passes through the three co-ordinate axes as

$$2fyz + 2gzx + 2hxy = 0 \text{ i.e. } fyz + gzx + hxy = 0.$$

Example 9.6 Find the equation of the cone which passes through the axes of co-ordinates and contains the points $(1, 1, 1)$ and $(1, -2, 1)$.
Solution. As the cone passes through the three co-ordinate axes , the vertex of the cone is at the origin.Hence the equation of the cone is of the form'

$$fyz + gzx + hxy = 0 \tag{9.13}$$

The cone passes through the points $(1, 1, 1)$ and $(-1, 2, 1)$. Co-ordinates of these two points satisfy the equation (9.13).

$$\therefore \quad f + g + h = 0 \quad and \quad 3f - g - 2h = 0$$

Solving the equations for f, g and h ,we get

$$\frac{f}{-1} = \frac{g}{4} = \frac{h}{-3} = k(say)$$
$$\therefore \quad f = -k, \quad g = 4k, \quad h = -3k.$$

Substituting the values of f, g and h in (9.13) we get the equation of the required cone as,

$$-kyz + 4kzx - 3kxy = 0 \text{ i.e. } yz - 4zx + 3xy = 0.$$

Example 9.7 Find the equation of the cone passing through the co-ordinate axes and having the lines, $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x}{3} = \frac{y}{-1} = \frac{z}{-1}$ as generators.
Solution. As the cone passes through the three co-ordinate axes, the vertex of the cone is at the origin. Hence the equation of the cone is of the form

$$fyz + gzx + hxy = 0 \tag{9.14}$$

Given that the lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3} \quad and \quad \frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ are the generators of the cone, thus $d.r.s$ of the lines satisfy the equation $(8.8.3)$

$$\begin{aligned} \therefore \quad f(-2)(3) + g(3)(1) + h(1)(-2) &= 0 \\ and \quad f(-1)(-1) + g(-1)(3) + h(3)(-1) &= 0 \end{aligned}$$

$$\therefore \quad -6f + 3g - 2h = 0 \quad and \quad f - 3g - 3h = 0.$$

Solving these two equations for f, g and h we get

$$\frac{f}{-1} = \frac{g}{4} = \frac{h}{-3} = k(say)$$
$$\therefore \quad f = -15k, \quad g = -20k, \quad h = 15k.$$

Substituting the values of f, g and h in $(8.8.3)$, we get the equation of the required cone as

$$-15kyz - 20kzx + 15kxy = 0 \text{ i.e. } \quad 3yz + 4zx + 3xy = 0.$$

Example 9.8 Show that the line $\frac{x}{2} = \frac{y}{-1} = \frac{z}{3}$ is a generator of the cone $x^2 + y^2 + z^2 + 4xy - xz = 0$.
Solution. The equation of the cone is,

$$x^2 + y^2 + z^2 + 4xy - xz = 0 \tag{9.15}$$

which is a homogenous equation. Hence the vertex of the cone is the origin. If we show that $d.r.s$ of the line $\frac{x}{2} = \frac{y}{-1} = \frac{z}{3}$ satisfy the equation (9.15), then we can say that the given line is a generator of the cone given by the equation (9.15). $d.r.s$ of the line $\frac{x}{2} = \frac{y}{-1} = \frac{z}{3}$ are $2, -1, 3$. Substituting these values in $L.H.S.$ of (9.15) we get,

$$2^2 + (-1)^2 + 3^2 + 4(2)(-1) - (2)(3).$$

Which is equal to 0. Thus, the line $\frac{x}{2} = \frac{y}{-1} = \frac{z}{3}$ is the generator of the given cone.

Example 9.9 Show that the equation $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$ represents a cone with with vertex at the point $(-1, -2, -3)$
Solution.

$$4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0 \tag{9.16}$$

Shift the origin to the point $(-1, -2, -3)$. Let (x, y, z) and (x', y', z') be respectively the old and new co-ordinates of the points on the cone. Then $x = x' - 1, y = y' - 1, z = z' - 1$ Substituting in to the equation (9.16) we get

$$\begin{aligned} 4(x'-1)^2 - (y'-2)^2 + 2(z'-3)^2 + 2(x'-1)(y'-2) \\ -3(y'-2)(z'-3) + 12(x'-1) - 11(y'-2) + 6(z'-3) + 4 = 0 \end{aligned}$$

On simplification gives

$$4x'^2 - y'^2 + 2z'^2 + 2x'y' - 3y'z' = 0 \tag{9.17}$$

The equation (9.17) is a homogeneous equation in x', y', z'. Hence the equation (9.17) represents a cone with vertex at origin in the new co-ordinate system. Thus the equation (9.16) represents a cone with vertex at $(-1, -2, -3)$.

Example 9.10 Find the equation of the cone whose vertex is at origin and the guiding curve is a circle $y^2 + z^2 = 16, x = 2$. Show that section of the cone by the plane $z = 1$ is a hyperbola.
Solution. Since the vertex of the cone is at origin, the equation of the cone is a homogeneous equation in x, y, z. Consider the equation of the guiding curve $y^2 + z^2 = 16, x = 2$. We make one of the equations homogeneous with the help of the other. We make $y^2 + z^2 = 16$ homogeneous with the help of $x = 2$

$$\therefore \quad y^2 + z^2 = 16 \times 1^2 \therefore \quad y^2 + z^2 = 16 \times \left(\frac{x}{2}\right)^2 \quad , since \quad \frac{x}{2} = 1$$

Hence the equation of the cone is $4x^2 - y^2 - z^2 = 0$. Now the section of the cone by the plane $z = 1$ is,

$$4x^2 - y^2 - 1^2 = 0, \quad z = 1 \text{ i.e. } \quad 4x^2 - y^2 = 1, \quad z = 1$$

which is a hyperbola.

Example 9.11 Find the equation of the cone with its vertex at the origin and whose guiding curve is given by $x^2+y^2+z^2-2x+2y+4z-3=0$, $x^2+y^2+z^2+2x+4y+6z-11=0$.
Solution. Since the vertex of the cone is at origin, the equation of the cone is a homogeneous equation in x, y, z. The equation of the cone is obtained by making the equation

$$x^2+y^2+z^2-2x+2y+4z-3=0 \tag{9.18}$$

homogeneous with the help of the equation

$$x^2+y^2+z^2+2x+4y+6z-11=0 \tag{9.19}$$

Subtracting the equation (9.18) from the equation (9.19) we get

$$4x+2y+2z-8=0 \quad i.e. \quad \frac{2x+y+z}{4}=1 \tag{9.20}$$

using (9.20) we make the equation (9.17) homogeneous as follows,

$$x^2+y^2+z^2-2x\left(\frac{2x+y+z}{4}\right)$$
$$+2y\left(\frac{2x+y+z}{4}\right)+4z\left(\frac{2x+y+z}{4}\right)-3\left(\frac{2x+y+z}{4}\right)=0$$

Simplification yields,

$$12x^2-21y^2-29z^2-18yz-12zx+4xy=0$$

which is the required equation of the cone.

Example 9.12 Find the equation of the right circular cone which passes through the point $(1,-2,3)$ whose vertex is at $(2,-3,5)$ and whose axis makes equal angles with co-ordinate axes.
Solution. We are given that axis of the right circular cone makes equal angles with co-ordinate axes .Therefore the $d.r.s$ of the axis are $1,1,1$. Let

$A(1,-2,3)$ and $V(2,-3,5)$. Therefore the $d.r.s$ of VA are $2-1, -3+2, 5-3$ i.e.$1,-1$ and 2 respectively. Let θ be the semi vertical angle. The θ is the angle between the axis of the cone and VA

$$\therefore \quad \cos\theta=\frac{1(1)+1(-1)+1(2)}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-1)^2+2^2}}=\frac{2}{\sqrt{3}\sqrt{6}} \tag{9.21}$$

Let $P(x,y,z)$ be a point on the right circular cone.Then VP is a generator and its $d.r.s$ are $x-2, y+3, z-5$. Then θ is the angle between VP and the axis

$$\therefore \quad \cos\theta=\frac{1(x-2)+1(y+3)+1(z-5)}{\sqrt{1^2+1^2+1^2}\sqrt{(x-2)^2+(y-3)^2+(z-5)^2}} \tag{9.22}$$

From (9.21) and (9.22) we get

$$\frac{2}{\sqrt{3}\sqrt{6}}=\frac{x+y+z-4}{\sqrt{3}\sqrt{(x-2)^2+(y-3)^2+(z-5)^2}}$$
$$i.e. \quad 2[(x-2)^2+(y-3)^2+(z-5)^2]^2=3(x+y+z-4)^2,$$

which is the required equation of the right circular cone.

Example 9.13 Find the equation of the right circular cylinder of radius 2, whose axis passes through $(1,2,3)$ and has $d.c.s$ proportional to $2,-3,6$.
Solution. Let $A(1,2,3)$ and $P(x,y,z)$ be a point on the cylinder. Draw PM perpendicular to the axis of the cylinder. Then PM is the radius of the cylinder so $PM=2$.. By Distance formula, $AP^2=(x-1)^2+(y-2)^2+(z-3)^2$. Let MA pbe the projection of AP on the axis. $\therefore$ $MA=\frac{2(x-1)-3(y-2)+6(z-3)}{\sqrt{2^2+(-3)^2+6^2}}$ as d.r.s of the axis are $2,-3,6$. Thus, $MA=\dfrac{2x-3y+6z-14}{7}$.

Now from the right angled $\triangle AMP$, we get $AP^2-MA^2=9$

$$\left((x-1)^2+(y-2)^2+(z-3)^2\right)-\left(\frac{2x-3y+6z-14}{7}\right)^2=9$$

Simplification of the equation gives the required equation of the right circular cylinder as

$$45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0.$$

9.9 Exercise

1. Find the equation of a cone whose vertex is at $(-1, 1, 2)$ and guiding curve is $3x^2 - y^2 = 1; z = 0$.

2. Find the equation of a cone with vertex at the origin and containing the curve $x^2 + y^2 = 4; z = 5$.

3. Find the equation of a cone whose vertex is at $(1, 1, 3)$ and passing through $4x^2 + z^2 = 1; y = 4$.

4. The axis of a right circular cone with vertex at the origin makes equal angles with the coordinate axes. If the cone passes through the line drawn from the origin with direction rations $1, -2, 2$, find the equation of the cone.

5. Find the equation of the cylinder whose generators are parallel to the line $6x = -3y = 2z$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1; z = 3$.

6. Lines are drawn parallel to the line $\frac{x-3}{l} = \frac{y-4}{m} = \frac{z-5}{n}$ through the points on the circle $x^2+y^2 = a^2$ in $ZOX-plane$. Find the equation of the surface so formed.

7. Find the equation of the right circular cylinder of radius 2 and having as axis the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$.

8. Find the equation of the right circular cone having $P(2, -3, 5)$ as a vertex; axis PQ which makes equal angles with coordinate axes and the semi vertical angle is 30^o

9. Show that $x^2 + 2y^2 + z^2 - 4yz - 6zx - 2x + 8y - 2z + 9 = 0$ represents a cone with vertex at $(1, -2, 0)$.

10. Find the equation of a cone with vertex the origin and base a circle in the plane $z = 12$ with centre $(13, 0, 12)$ and radius 5.Also show that the section of any plane parallel to $x = 0$ is a circle.

11. Find the equation of the right circular cylinder of radius 2 whose axis passes through$(1, 2, 3)$ and has d.r.s. $2, -3, 6$.

12. Find the equation of a cone whose vertex is at the origin and direction ratios of whose generators satisfy the equation $3l^2-2m^2+5n^2 = 0$.

13. The equation of a cone is $x^2 + 2y^2 + z^2 - 2yz + zx - 3xy = 0$. Test whether the following lines are generators of the cone.
(a) $x = -y = z$ (b) $x = y = z$ (c) $\frac{x}{2} = \frac{y}{3} = \frac{z}{2}$ (d) $\frac{x}{3} = \frac{y}{-1} = \frac{z}{2}$.

14. Find the equation of a cone with vertex at the origin and which passes through the curve $x^2 + y^2 + z^2 + x - 2y + 3z = 4; x^2 + y^2 + z^2 + 2x - 3y + 4z = 5$.

15. Find the equation of the right circular cylinder of radius 2 whose axis lies along the line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$.

16. Obtain the equation of the right circular cylinder whose guiding curve is the circle $x^2 + y^2 + z^2 - 9 = 0; x - y + z - 3 = 0$.

17. Lines are drawn through the origin having direction ratios $1, 2, 2$; $2, 3, 6$; and $3, 4, 12$. Show that the axis of the right circular cone through them has d.c.s.$\frac{-1}{\sqrt{3}}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}$ and the semi vertical angle of the cone is —-. Also obtain the equation of the cone.

18. Show that the equation $2y^2-8yz-4zx-8xy+6x-4y-2z+5 = 0$ represents a cone whose vertex is $(\frac{-7}{6}, \frac{1}{3}, \frac{5}{6})$.

19. Determine the equation of the right circular cone having vertex at $(2, 3, 1)$, axis parallel to the line $2x = -y = -2z$ and one of its generators having d.r.s. $1, 1, 1$.

20. Find the equation of the right circular cone generated by the lines drawn from the origin to cut the circle through the points $(1, 2, 2)$, $(2, 1, -2)$ and $(2, -2, 1)$.

21. Find the equation of the cone with vertex at the origin and containing the curve $ax^2 + by^2 = 2z; lx + my + nz = p$.

22. Obtain the equation of the right circular cone which is generated by revolving the line whose equations are $3x - y + z = 1; 5x + y + 3z + 1 = 0$ about the y-axis.

23. Find the equation of the cone which passes through the coordinate axes and has two generators having direction ratios 1,2,2 and -2,-2,1.

24. Obtain the equation of the cone which passes through the coordinate axes and has the lines $\frac{x}{2} = \frac{y}{1} = \frac{z}{3}$ and $\frac{x}{-3} = \frac{y}{1} = \frac{z}{-2}$ as its generators.

9.10 Answers

(1) $12x^2 - 4y^2 + z^2 + 4yz + 12zx + 4z - 4 = 0.$
(2) $25(x^2 + y^2) - 4z^2 = 0.$
(3) $12x^2 + 4y^2 + 3z^2 + 6yz + 8xy - 32x - 34y - 24z + 69 = 0.$
(4) $4x^2 + 4y^2 + 4z^2 + 9yz + 9zx + 9xy = 0.$
(5) $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0.$
(6) $(mx - ly)^2$+$(mz - ny)^2 = m^2a^2.$
(7) $5x^2 + 8y^2 + 5z^2 - 4yz - 8zx - 4xy + 22x - 16y - 14z + 26 = 0.$
(8) $5x^2 + 5y^2 + 5z^2 - 8yz - 8zx - 8xy + 8x + 86y + 278 = 0.$
(10) $6x^2 + 6y^2 + 6z^2 - 13xz = 0.$
(11) $9(2y + z - 7)^2 + 4(z - 3x)^2 + (3x + 2y - 7)^2 = 196.$
(12) $3x^2 - 2y^2 + 5z^2 = 0.$
(13) (b) and (c) are generators;(a) and (d) are not generators.
(14) $x^2 + y^2 - z^2 = 0.$
(15) $2x^2 + y^2 - 3yz + 4zx - 5xy = 0.$
(16) $26x^2 + 29y^2 + 5z^2 + 10yz - 24zx - 4xy + 150y + 30z + 75 = 0.$
(17) $x^2 + y^2 + z^2 - zx + xy = 0.$ (18) $yz - zx - xy = 0.$
(20) $x^2 - 8y^2 - z^2 - 12yz + 6zx + 12xy - 46x + 38y + 22z - 19 = 0.$
(21) $8x^2 - 4y^2 - 4z^2 + yz + 5zx + 5xy = 0.$
(22) $apx^2 + bpy^2 - 2nz^2 - 2myz - 2lzx = 0.$
(23) $x^2 - 5y^2 + z^2 - 10y - 5 = 0.$
(24) $yz - zx - xy = 0.$ (25) $6yz - zx - 6xy = 0.$