Chapter-1

Limit And Continuity

* **Function of two variables :-** A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D, a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on i.e. $\{f(x,y) \mid (x,y) \in D\}$ we write it as $z = f(x, y)$ Here the variables x & y are independent variables and z is the dependent variable.

$y = f(x)$ ↓ indep. variable ↓ dep. variable

* **Note :-** If a function is given by formula and no domain is specified, then the domain of f is understood to be the set of all pair (x, y) for which the given expression is a well-defined real numbers.

ex ①: For each of the following function, evaluate $f(3, 2)$ and find and sketch the domain.

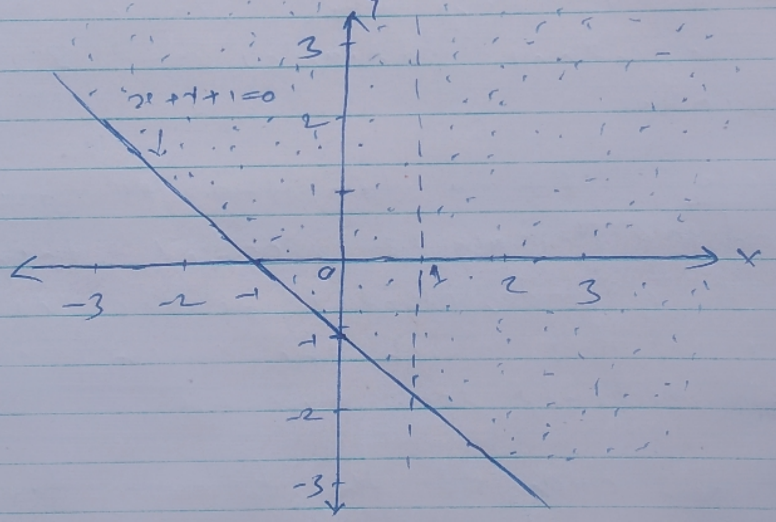
ⓐ $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$ ⓑ $f(x,y) = x \log(y^2 - x)$.

Sol^n:- ⓐ $f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$.

The value of f exists only if the denominator is not zero and the quantity under the square root sign is non-negative so the domain of f is $D = \{(x,y) \mid x+y+1 \geq 0,\ x \neq 1\}$

The equation $x+y+1=0 \Rightarrow y=-x-1$ is the equation of line

$\therefore$ The inequality $x+y+1 \geq 0$ ie. $y \geq -x-1$ shows that the points that lie on or above the line $y=-x-1$. Also $x \neq 1$ indicates that we have to remove the points on the line $x=1$ from the domain.

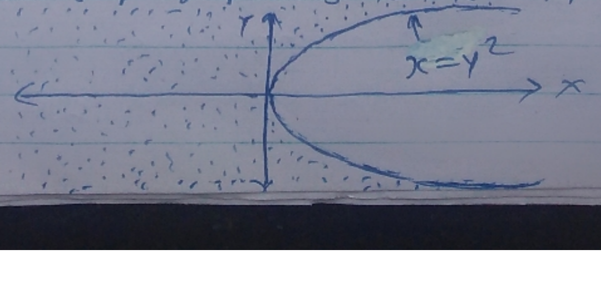


(b) $f(3,2) = 3\log(2^2-3) = 3\log 1 = 3\cdot 0 = 0$

Since $\log(y^2-x)$ is defined only if $y^2-x>0$ ie. $x<y^2$ so the domain of f is

$$D = \{(x,y) \mid x<y^2\}$$

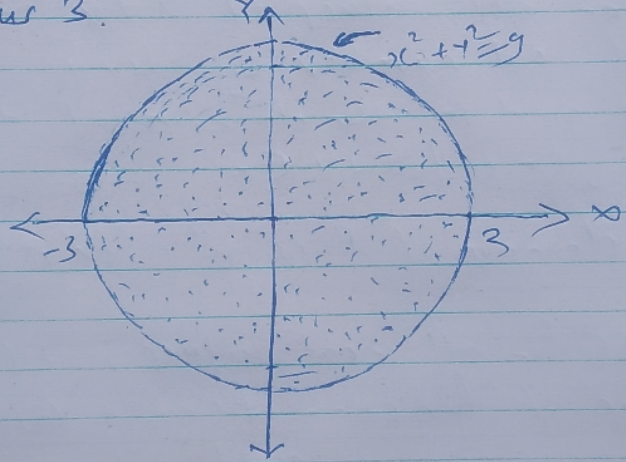
ie. the set of points to the left of the parabola $x=y^2$



ex(2): Find the domain and range of
$f(x,y) = \sqrt{9-x^2-y^2}$

Sol^n:- The domain of f is $D = \{(x,y) / 9-x^2-y^2 \geq 0\}$
$= \{(x,y) / x^2+y^2 \leq 9\}$

which is the disk with center $(0,0)$ and radius 3.



The range of f is $\{z / z = \sqrt{9-x^2-y^2},\ (x,y) \in D\}$

Since z is a positive square root, we have

$z \geq 0 \Rightarrow 0 \leq z$ — (1)

Also $x^2+y^2 \geq 0$

$\{\because$ The expression is sum of square of two quantity $\}$

$\Rightarrow -x^2-y^2 \leq 0$

$\Rightarrow 9-x^2-y^2 \leq 9$

$\Rightarrow \sqrt{9-x^2-y^2} \leq 3$

$\Rightarrow z \leq 3$ — (2)

~~from eq^n (1) & (2) we have~~

$0 \leq z \leq 3$

So the range of f is $\{z / 0 \leq z \leq 3\} = [0,3]$.

ex. 3: Let $f(x, y) = \cos(x + 2y)$. Evaluate $f(2, -1)$. Also, find the domain and range of f.

Sol^n: Let $f(x, y) = \cos(x + 2y)$

$$\Rightarrow f(2, -1) = \cos(2 + 2(-1)) = \cos 0 = 1$$

As for all values of x & y in the set of real number R, $f(x, y) = \cos(x + 2y)$ exists and so the domain of f is the set $\{(x, y) \mid x, y \in R\}$

$\therefore$ The domain of f is $R^2 = \{(x, y) \mid x, y \in R\}$

As for all real values of x & y, the value of function $f(x, y) = \cos(x + 2y)$ lies between -1 and $+1$

$\left\{ \begin{array}{l} \because \text{All the values of trignometric function} \\ \text{sin \& cos lies between } -1 \text{ \& } 1 \end{array} \right\}$

And so for the function $z = f(x, y) = \cos(x + 2y)$ we have $-1 \leq z \leq 1$

$\therefore$ The range is $\{z \mid -1 \leq z \leq 1\} = [-1, 1]$

ex 4: Let $f(x, y) = 1 + \sqrt{5 - y^2}$. Evaluate $f(3, 1)$. Also, find the domain and the range of f

Sol^n: Let $z = f(x, y) = 1 + \sqrt{4 - y^2}$

$\Rightarrow f(3\ 1) = 1 + \sqrt{4-(1)^2}$

$= 1 + \sqrt{4-1}$

$= 1 + \sqrt{3}$

To find the domain of f:

The function $z = f(x\,y) = 1 + \sqrt{4-y^2}$ exists only if $4-y^2 \geq 0 \Rightarrow 4 \geq y^2$

$\Rightarrow y^2 \leq 4 \quad \Rightarrow \pm y \leq 2$

$\Rightarrow y \leq 2 \;\&\; -y \leq 2$

$\Rightarrow y \leq 2 \;\&\; y \geq -2$

$\Rightarrow y \leq 2 \;\&\; -2 \leq y$

$\Rightarrow -2 \leq y \leq 2$

$\therefore$ Domain of f is $D = \{(x\,y) / x \in R,\ -2 \leq y \leq 2\}$

$= [-2\ 2]$

To find Range of f:

For $y \in [-2\ 2]$, the value of $z = f(x\,y)$

$= 1 + \sqrt{4-y^2}$

belongs to $[1\ 3]$

$\left\{ \begin{array}{ll} \because \text{ for } y = -2 \;\&\; 2, & z = 1 + \sqrt{4-4} = 1 \\ \text{for } y = -1 \;\&\; 1, & z = 1 + \sqrt{4-1} = 1 + \sqrt{3} \\ \& \text{ for } y = 0, & z = 1 + \sqrt{4} = 3 \end{array} \right\}$

$\therefore$ For $z = f(x\,y) = 1 + \sqrt{4-y^2}$ we have $1 \leq z \leq 3$

$\therefore$ The range of f is $\{z / 1 \leq z \leq 3\} = [1\ 3]$